

PSC KÄHLER CONES OF MINIMAL KÄHLER SURFACES AND A BIRATIONAL OBSTRUCTION

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ABSTRACT. We characterize the positive scalar curvature Kähler cone of every minimal compact Kähler surface with Kodaira dimension $-\infty$. By the Enriques–Kodaira classification, every such surface is either $\mathbb{P}^2$ or a geometrically ruled surface $\mathbb{P}(E) \rightarrow \Sigma_g$, where E is a rank-two holomorphic vector bundle over a compact Riemann surface of genus g . On $\mathbb{P}^2$, every Kähler class admits a Kähler metric of positive scalar curvature, whereas on $\mathbb{P}(E)$, every Kähler class of positive total scalar curvature admits a metric of positive scalar curvature if and only if $g \leq 1$ or E is slope-semistable. We further prove that if a smooth compact Kähler surface admits a birational morphism onto a ruled surface $\mathbb{P}(E) \rightarrow \Sigma_g$, where $g \geq 2$ and E is slope-unstable, then it contains a Kähler class of positive total scalar curvature admitting no positive scalar curvature Kähler metric.

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1. INTRODUCTION

The existence problem for Riemannian metrics of positive scalar curvature (PSC) admits several criteria and obstructions. In Kähler geometry, metric existence problems are naturally formulated within a fixed Kähler class. Thus, even when the underlying complex manifold admits some Kähler metric of positive scalar curvature, it remains a separate problem to decide which Kähler classes contain such metrics.

Let X be a compact Kähler manifold and let $\mathcal{K}_X$ denote the Kähler cone. We define the PSC Kähler cone by

$$\mathcal{K}_X^{\text{psc}} := \{ \alpha \in \mathcal{K}_X : \exists \omega \in \alpha, \text{ s.t. } S(\omega) > 0 \}.$$

A necessary condition for a Kähler class $\alpha \in \mathcal{K}_X^{\text{psc}}$ is the positive total scalar curvature:

$$\int_X S(\omega) \omega^n = 2n\pi c_1(X) \cdot \alpha^{n-1} > 0.$$

Accordingly, one can define the positive total scalar curvature Kähler cone

$$\mathcal{T}_X^+ := \{ \alpha \in \mathcal{K}_X : c_1(X) \cdot \alpha^{n-1} > 0 \}.$$

It is clear that $\mathcal{K}_X^{\text{psc}} \subseteq \mathcal{T}_X^+$. Whether the reverse inclusion holds is an open question: an affirmative answer would give a purely numerical criterion for the existence of a positive scalar curvature Kähler metric in a prescribed Kähler class. Moreover, it would identify the positive scalar curvature cone, and in particular its boundary, by the intersection number $c_1(X) \cdot \alpha^{n-1} > 0$. A closely related question was asked by Brown in [Bro26, Question 7.6] also.

The purpose of this note is to show that the answer to the above question is negative in general. More precisely, we first characterize the PSC Kähler cone on minimal Kähler surfaces of Kodaira dimension $-\infty$. By the Enriques–Kodaira classification, such a surface is either $\mathbb{P}^2$, a Hirzebruch surface $\mathbb{F}_n$ with $n = 0$ or $n \geq 2$, or a geometrically ruled surface $\mathbb{P}(E) \rightarrow \Sigma_g$ over a compact Riemann surface of genus $g \geq 1$, where E is a rank-two holomorphic vector bundle, and $\mathbb{P}(E)$ denotes the projectivization of E ; see [BHPVdV04, Chapter VI].

On a minimal ruled surface $X = \mathbb{P}(E) \xrightarrow{\pi} \Sigma_g$, let $F := \pi^*[\text{pt}] \in H^{1,1}(X, \mathbb{R})$ denote the fibre class. Define the normalized tautological class (see [Miy87, p. 456]) by

$$\zeta := \frac{1}{2}c_1(K_{X/\Sigma_g}^{-1}) = c_1(\mathcal{O}_{\mathbb{P}(E)}(1)) - \frac{\deg E}{2} F.$$

Moreover, ζ depends only on the ruling and is unchanged under $E \mapsto E \otimes L$ for L a line bundle. In particular, F and ζ generate $H^{1,1}(X, \mathbb{R})$ and every real $(1, 1)$ -class can be written uniquely as $\alpha = 2\pi(a\zeta + bF)$.

For a nonzero holomorphic subbundle $S \subseteq E$, set $\mu(S) := \deg S / \text{rank } S$ and let $\mu_{\max}(E)$ denote the largest slope in the Harder–Narasimhan filtration of E . When E is semistable, we have $\mu_{\max}(E) = \mu(E)$. We then define

$$(1) \quad \Delta(E) := 2\mu_{\max}(E) - \deg E.$$

Thus $\Delta(E) \geq 0$, with equality if and only if E is slope-semistable. Moreover, $\Delta(E)$ is also unchanged under $E \mapsto E \otimes L$. Our first result is the following.

Theorem 1. *Let X be a minimal compact Kähler surface of Kodaira dimension $-\infty$.*

(i) *If $X = \mathbb{P}^2$ or $X = \mathbb{F}_n$ with $n = 0$ or $n \geq 2$, then*

$$\mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+ = \mathcal{K}_X.$$

(ii) *If $X = \mathbb{P}(E) \rightarrow \Sigma_g$ with $g \geq 1$, then*

$$\begin{aligned} \mathcal{T}_X^+ &= \left\{ 2\pi(a\zeta + bF) : a > 0, \quad b > \max \left\{ g - 1, \frac{\Delta(E)}{2} \right\} a \right\}, \\ \mathcal{K}_X^{\text{psc}} &= \left\{ 2\pi(a\zeta + bF) : a > 0, \quad b > \left(g - 1 + \frac{\Delta(E)}{2} \right) a \right\}. \end{aligned}$$

Consequently,

$$\mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+ \iff g = 1 \text{ or } E \text{ is slope-semistable.}$$

For decomposable bundles $E = \mathcal{O}_{\Sigma_g} \oplus L$, we also give in Appendix A a direct momentum construction, which therefore provides a concrete geometric model for the obstruction appearing in the above result.

The variational characterization in [Sha26, Theorem A] yields a general stability obstruction to equality of the two cones. We show moreover that this obstruction persists under birational morphisms. Applying this principle to the deformation of the normal cone of the Harder–Narasimhan section gives the following result.

Theorem 2. *Let X be a smooth compact Kähler surface. Suppose that X admits a birational morphism $q : X \rightarrow Y = \mathbb{P}(E)$, where E is a slope-unstable rank-two holomorphic vector bundle over a compact Riemann surface Σ_g of genus $g \geq 2$. Then*

$$\mathcal{K}_X^{\text{psc}} \subsetneq \mathcal{T}_X^+.$$

Theorem 2 gives a necessary birational condition for equality of the two cones. It is natural to ask whether this condition is also sufficient. Theorem 1 gives an affirmative answer when X is minimal. In general, the converse remains open.

These results suggest the following possible picture for compact Kähler surfaces (not necessarily minimal) of Kodaira dimension $-\infty$. Equality $\mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+$ should always hold for rational surfaces and for non-rational ruled surfaces based on elliptic curves. For non-rational ruled surfaces over a curve of genus at least two, one may ask whether equality is equivalent to the semistability of the rank-two bundle defining every minimal ruled surface dominated by X . In higher dimensions, it might be interesting to seek an analogous algebraic stability condition whose satisfaction is equivalent to $\mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+$.

Organization of the paper. The paper is organized as follows. Section 2 recalls the geometry of ruled surfaces and fixes the notation used throughout the paper. Section 3 develops the non-Archimedean prescribed scalar curvature measure functional and derives the obstruction associated with the deformation to the normal cone of the Harder–Narasimhan section for an unstable vector bundle E . Section 4 combines this obstruction with deformation arguments to determine the PSC Kähler cones of minimal surfaces and prove Theorem 1. Section 5 lifts the obstruction through point blowups and proves Theorem 2. Appendix A gives the momentum construction on decomposable ruled surfaces and identifies the resulting invariant cone with the full PSC Kähler cone.

On the use of AI. Proposition A.2 was suggested by ChatGPT 5.5 Plus as a key step of the counterexample when we were trying to *prove* $\mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+$ on every Kähler surface X . After that, we improved this mechanism and related it to the slope stability for ruled surfaces. We also used ChatGPT 5.5 Plus and 5.6 Sol as an assisting tool to refine the writing and presentation, and to help verify the mathematical logic.

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2. GEOMETRY OF RULED SURFACES

In this section, we introduce some fundamental facts of ruled surfaces and keep our notations. Standard references for ruled surfaces are [Har83, Chapter V, Section 2] and [BHPVdV04, Chapter V].

Let $X = \mathbb{P}(E) \xrightarrow{\pi} \Sigma_g$ be a geometrically ruled surface, and let F and ζ be the fibre class and the normalized tautological class introduced in Section 1. The intersection numbers are given by

$$(2) \quad \zeta^2 = 0, \quad \zeta \cdot F = 1, \quad F^2 = 0.$$

The first Chern class is then given by

$$(3) \quad c_1(X) = 2\zeta + (2 - 2g)F.$$

Consequently, if $A = a\zeta + bF$, then

$$(4) \quad A^2 = 2ab, \quad c_1(X) \cdot A = 2b + (2 - 2g)a.$$

Suppose that E is unstable. Then the Harder–Narasimhan filtration [HN75] has the form

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

where $\deg L = \mu_{\max}(E)$. The quotient $E \rightarrow Q$ determines a section $C \subset X$, called the *Harder–Narasimhan section*. Thus C is the section associated with the quotient of E by its maximal-slope subbundle; see also [LN83]. Its numerical class and self-intersection are

$$(5) \quad C \equiv \zeta - \frac{\Delta(E)}{2}F, \quad C^2 = -\Delta(E).$$

In the decomposable case $E = \mathcal{O}_{\Sigma_g} \oplus L$ with $\deg L = -d < 0$, one has $d = \Delta(E)$. We denote the Harder–Narasimhan section by C_0 induced, in the quotient convention, by $\mathcal{O}_{\Sigma_g} \oplus L \rightarrow L$. It satisfies

$$C_0^2 = -d, \quad C_0 = \zeta - \frac{d}{2}F.$$

The standard numerical description of the Kähler cone is

$$(6) \quad A = a\zeta + bF \in \mathcal{K}_X \iff a > 0, \quad b > \frac{\Delta(E)}{2}a.$$

Indeed, the description of the nef cone of a projective bundle over a curve originates from Miyaoka [Miy87] (for a more precise version, one can check [Ful11, Lemma 2.1]). Therefore, (6) follows from (2), (5), and the Nakai–Moishezon criterion. Combining (4) and (6), we obtain

$$(7) \quad \mathcal{T}_X^+ = \left\{ 2\pi(a\zeta + bF) : a > 0, \quad b > \max \left\{ g - 1, \frac{\Delta(E)}{2} \right\} a \right\}.$$

3. THE PSC KÄHLER CONE OF UNSTABLE MINIMAL RULED SURFACES

In this section, we determine explicitly the PSC Kähler cone of a ruled surface $\mathbb{P}(E) \rightarrow \Sigma_g$ where E is unstable. The necessary numerical condition is obtained by combining the variational characterization of PSC Kähler metrics with the non-Archimedean slope of the deformation to the normal cone of the Harder–Narasimhan section.

3.1. The non-Archimedean $\mathcal{F}$ -functional. Let Z be a smooth projective surface and let A be a $\mathbb{Q}$ -ample class satisfying $c_1(Z) \cdot A > 0$. Set $\alpha = 2\pi A$, $V_\alpha = \alpha^2$, and fix a reference Kähler form $\omega \in \alpha$. Since

$$\bar{S}_\alpha = \frac{4\pi c_1(Z) \cdot \alpha}{\alpha^2} = \frac{2c_1(Z) \cdot A}{A^2} > 0,$$

the measure $\Omega = \bar{S}_\alpha \omega^2$ belongs to the admissible class considered in [Sha26]. For $\phi \in \mathcal{H}_\omega$, the corresponding prescribed scalar-curvature measure functional is

$$\begin{aligned} \mathcal{F}_{\omega, \Omega}(\phi) &= \mathcal{M}_\omega(\phi) - \bar{S}_\alpha E_\omega(\phi) + \int_Z \phi \Omega \\ &= \mathcal{M}_\omega(\phi) + \bar{S}_\alpha \left(\int_Z \phi \omega^2 - E_\omega(\phi) \right), \end{aligned}$$

where $\mathcal{M}_\omega$ is the Mabuchi K -energy and E_ω is the Aubin–Yau energy. After dividing both sides by $2V_\alpha$, the second term is the Aubin J -functional, and we denote these normalized functionals by

$$M(\phi) := \frac{\mathcal{M}_\omega(\phi)}{2V_\alpha}, \quad J(\phi) := \frac{1}{2V_\alpha} \left(\int_Z \phi \omega^2 - E_\omega(\phi) \right), \quad \mathcal{F}(\phi) := \frac{\mathcal{F}_{\omega, \Omega}(\phi)}{2V_\alpha}.$$

Thus we can simply write the $\mathcal{F}$ -functional as

$$(8) \quad \mathcal{F} = M + \bar{S}_\alpha J.$$

Since A is rational, we identify it with the numerical class of an ample $\mathbb{Q}$ -line bundle on Z . Recall that a normal ample test configuration for (Z, A) consists of a normal variety $\mathcal{Z}$, a flat projective morphism $\pi : \mathcal{Z} \rightarrow \mathbb{A}^1$, a $\mathbb{C}^*$ -action on $\mathcal{Z}$ lifting the standard action on $\mathbb{A}^1$, and a $\mathbb{C}^*$ -linearized relatively ample $\mathbb{Q}$ -line bundle $\mathcal{A}$, together with an equivariant identification

$$(\mathcal{Z}, \mathcal{A})|_{\mathbb{C}^*} \simeq (Z, A) \times \mathbb{C}^*;$$

see [BHJ17, Definitions 2.1 and 2.2]. For such a test configuration, we define

$$(9) \quad \mathcal{F}^{\text{NA}}(\mathcal{Z}, \mathcal{A}) := M^{\text{NA}}(\mathcal{Z}, \mathcal{A}) + \overline{S}_\alpha J^{\text{NA}}(\mathcal{Z}, \mathcal{A}),$$

where M^{NA} and J^{NA} are the non-Archimedean Mabuchi K -energy and Aubin J -functionals of [BHJ17, Definitions 7.13 and 7.6].

Lemma 3.1. *If α contains a Kähler metric of positive scalar curvature, then there exists a constant $\delta > 0$ such that*

$$(10) \quad \mathcal{F}^{\text{NA}}(\mathcal{Z}, \mathcal{A}) \geq \delta J^{\text{NA}}(\mathcal{Z}, \mathcal{A})$$

for every normal ample test configuration $(\mathcal{Z}, \mathcal{A})$ for (Z, A) .

Proof. Since α contains a PSC Kähler metric, by [Sha26, Theorem A], the $\mathcal{F}$ -functional is d_1 -coercive on $\mathcal{E}_0^1(Z, \omega)$. Thus there exist constants $\varepsilon > 0$ and $C > 0$ such that

$$\mathcal{F}(\varphi) \geq \varepsilon d_1(0, \varphi) - C.$$

On $\mathcal{E}_0^1(Z, \omega)$, [Dar15, Eq. (62)] gives a constant $C_J > 0$ such that

$$d_1(0, \varphi) \geq C_J^{-1} J(\varphi).$$

Consequently,

$$(11) \quad \mathcal{F}(\varphi) \geq \varepsilon C_J^{-1} J(\varphi) - C.$$

Let $(\mathcal{Z}, \mathcal{A})$ be a normal ample test configuration, and let φ_s be the ray induced by a smooth strictly positive S^1 -invariant metric on $\mathcal{A}$ near the central fibre. Set $\widehat{\varphi}_s := \varphi_s - E_\omega(\varphi_s)/V_\alpha$. Then $\widehat{\varphi}_s \in \mathcal{E}_0^1(Z, \omega)$. Since $\mathcal{F}$ and J are invariant under addition of constants,

$$\mathcal{F}(\widehat{\varphi}_s) = \mathcal{F}(\varphi_s), \quad J(\widehat{\varphi}_s) = J(\varphi_s).$$

Applying (11), dividing by s , and letting $s \rightarrow +\infty$, the slope formulas [BHJ19, Theorem A] give

$$\mathcal{F}^{\text{NA}}(\mathcal{Z}, \mathcal{A}) \geq \varepsilon C_J^{-1} J^{\text{NA}}(\mathcal{Z}, \mathcal{A}).$$

This proves the assertion with $\delta = \varepsilon C_J^{-1}$. $\square$

Let $\pi : (\mathcal{Z}, \mathcal{A}) \rightarrow \mathbb{A}^1$ be a normal ample test configuration for (Z, A) . The compactification $\overline{\pi} : (\overline{\mathcal{Z}}, \overline{\mathcal{A}}) \rightarrow \mathbb{P}^1$ is obtained by gluing $(\mathcal{Z}, \mathcal{A})$ to $(Z, A) \times (\mathbb{P}^1 \setminus \{0\})$ along their common restriction over $\mathbb{C}^*$; see [BHJ17, Definition 2.7]. Thus $\overline{\mathcal{Z}}$ is a normal threefold, $\overline{\mathcal{A}}$ is the resulting relatively ample $\mathbb{Q}$ -line bundle, and

$$(\overline{\mathcal{Z}}, \overline{\mathcal{A}})|_{\mathbb{P}^1 \setminus \{0\}} \simeq (Z, A) \times (\mathbb{P}^1 \setminus \{0\}).$$

For the test configurations considered below, the birational map $\overline{\mathcal{Z}} \dashrightarrow Z \times \mathbb{P}^1$ induced by this identification extends to a $\mathbb{C}^*$ -equivariant morphism $\rho : \overline{\mathcal{Z}} \rightarrow Z \times \mathbb{P}^1$ over $\mathbb{P}^1$. In this case, we say that $\overline{\mathcal{Z}}$ dominates the trivial compactification. Let $p_Z : Z \times \mathbb{P}^1 \rightarrow Z$, $p_{\mathbb{P}^1} : Z \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the two projections. Then $\overline{\pi} = p_{\mathbb{P}^1} \circ \rho$, and we set $H := \rho^* p_Z^* A$.

Let $\mathcal{Z}_0 = \bar{\pi}^{-1}(0)$ denote the scheme-theoretic central fibre. Replacing $\bar{\mathcal{A}}$ by $\bar{\mathcal{A}} + q\mathcal{Z}_0$, for $q \in \mathbb{Q}$, corresponds to translating the associated non-Archimedean metric. Since J^{NA} and $\mathcal{F}^{\text{NA}}$ are translation invariant, and since $\mathcal{Z}_0 \cdot H^2 = A^2$, we may translate $\bar{\mathcal{A}}$ so that

$$(12) \quad \bar{\mathcal{A}} \cdot H^2 = 0.$$

Since $\dim Z = 2$, by [BHJ17, Definition 6.11], the non-Archimedean Aubin-Yau energy is defined by

$$E^{\text{NA}}(\mathcal{Z}, \mathcal{A}) := \frac{\bar{\mathcal{A}}^3}{3A^2}.$$

Moreover, [BHJ17, Definitions 7.2 and 7.6] gives

$$J^{\text{NA}}(\mathcal{Z}, \mathcal{A}) = \frac{\bar{\mathcal{A}} \cdot H^2}{A^2} - E^{\text{NA}}(\mathcal{Z}, \mathcal{A}).$$

Define the relative log canonical divisor by

$$(13) \quad K_{\bar{\mathcal{Z}}/\mathbb{P}^1}^{\log} := K_{\bar{\mathcal{Z}}/\mathbb{P}^1} - (\mathcal{Z}_0 - \mathcal{Z}_{0,\text{red}}).$$

By [BHJ17, Definition 7.13],

$$M^{\text{NA}}(\mathcal{Z}, \mathcal{A}) = \frac{K_{\bar{\mathcal{Z}}/\mathbb{P}^1}^{\log} \cdot \bar{\mathcal{A}}^2}{A^2} + \bar{S}_\alpha E^{\text{NA}}(\mathcal{Z}, \mathcal{A}).$$

Combining these formulas with (9) and (12), we obtain

$$(14) \quad \mathcal{F}^{\text{NA}}(\mathcal{Z}, \mathcal{A}) = \frac{K_{\bar{\mathcal{Z}}/\mathbb{P}^1}^{\log} \cdot \bar{\mathcal{A}}^2}{A^2}, \quad J^{\text{NA}}(\mathcal{Z}, \mathcal{A}) = -\frac{\bar{\mathcal{A}}^3}{3A^2}.$$

If the central fibre is reduced scheme-theoretically, then $\mathcal{Z}_0 = \mathcal{Z}_{0,\text{red}}$, and hence $K_{\bar{\mathcal{Z}}/\mathbb{P}^1}^{\log} = K_{\bar{\mathcal{Z}}/\mathbb{P}^1}$.

3.2. Deformation to the normal cone. Let $X = \mathbb{P}(E) \rightarrow \Sigma_g$, for $g \geq 1$, where E is unstable, and let C be the Harder–Narasimhan section.

Proposition 3.2. *If the Kähler class $2\pi A = 2\pi(a\zeta + bF)$ contains a PSC Kähler metric, then*

$$(15) \quad 2b > (\Delta(E) + 2g - 2)a.$$

Proof. Assume first that A is rational. Consider the compactified deformation to the normal cone

$$(16) \quad \rho : \bar{\mathcal{X}} = \text{Bl}_{C \times \{0\}}(X \times \mathbb{P}^1) \longrightarrow X \times \mathbb{P}^1,$$

and denote its exceptional divisor by P . Set

$$(17) \quad H = \rho^* p_X^* A, \quad \bar{\mathcal{A}}_c = H - cP.$$

The Seshadri constant of C with respect to A is a . Indeed,

$$A - cC = (a - c)\zeta + \left(b + \frac{c\Delta(E)}{2}\right)F$$

is Kähler for $0 < c < a$, while for $c > a$, $(A - cC) \cdot F < 0$. It follows from [RT06, Definition 3.2 and Lemma 4.1] that $\bar{\mathcal{A}}_c$ is relatively ample for every rational $c \in (0, a)$. Thus $(\bar{\mathcal{X}}, \bar{\mathcal{A}}_c)$ is the compactification of an ample test configuration for (X, A) . Moreover, $\bar{\mathcal{A}}_c \cdot H^2 = 0$, so the normalization (12) is automatic.

The total space is smooth and its central fibre is reduced. Since the blowup center has codimension two,

$$K_{\bar{\mathcal{X}}/\mathbb{P}^1} = \rho^* p_X^* K_X + P.$$

The intersection numbers give

$$H^2 \cdot P = 0, \quad H \cdot P^2 = -A \cdot C, \quad \rho^* p_X^* K_X \cdot P^2 = -K_X \cdot C, \quad P^3 = -\deg N_{C/X} = \Delta(E).$$

By the adjunction formula

$$K_X \cdot C = 2g - 2 - C^2 = 2g - 2 + \Delta(E).$$

Therefore, we obtain

$$(18) \quad K_{\overline{X}/\mathbb{P}^1} \cdot \overline{\mathcal{A}}_c^2 = 2cA \cdot C + (2 - 2g)c^2, \quad \overline{\mathcal{A}}_c^3 = -3c^2A \cdot C - c^3\Delta(E).$$

Equation (14) therefore yields, after regarding $\mathcal{F}^{\text{NA}}, J^{\text{NA}}$ as functions depending on c , that

$$(19) \quad \mathcal{F}^{\text{NA}}(c) = \frac{2cA \cdot C + (2 - 2g)c^2}{A^2}, \quad J^{\text{NA}}(c) = \frac{c^2}{A^2} \left(A \cdot C + \frac{\Delta(E)}{3}c \right).$$

Since A is Kähler, $A \cdot C > 0$, and hence

$$\lim_{c \nearrow a} J^{\text{NA}}(c) > 0.$$

If $2\pi A$ contains a PSC Kähler metric, apply Lemma 3.1 for rational $c \in (0, a)$ and let $c \nearrow a$. By (19),

$$2aA \cdot C + (2 - 2g)a^2 > 0.$$

Since $A \cdot C = b - \frac{\Delta(E)}{2}a$, we obtain (15) for rational class.

Now let $A = a\zeta + bF$ be an arbitrary real Kähler class such that $2\pi A$ contains a PSC Kähler metric ω . Then there exists a neighborhood U of A such that, for a', b' sufficiently close to a, b , $A' = a'\zeta + b'F \in U$ is still Kähler and contains a PSC Kähler metric. Since rational numbers are dense in real numbers, we can choose $a'_i, b'_i \in \mathbb{Q}$ such that $a'_i \rightarrow a, b'_i \rightarrow b$ as $i \rightarrow +\infty$. Hence,

$$2b - (\Delta(E) + 2g - 2)a = \lim_{i \rightarrow +\infty} (2b'_i - (\Delta(E) + 2g - 2)a'_i) \geq 0.$$

Suppose that $2b = (\Delta(E) + 2g - 2)a$. Then we can choose $\tilde{a}, \tilde{b} \in \mathbb{Q}$ sufficiently close to a, b with $\tilde{b} < b$ and $\tilde{a} > a$, so that $2\pi(\tilde{a}\zeta + \tilde{b}F)$ is Kähler and has positive scalar curvature. Clearly, $2\tilde{b} - (\Delta(E) + 2g - 2)\tilde{a} < 0$ which contradicts (15) for rational class. Hence

$$2b - (\Delta(E) + 2g - 2)a > 0,$$

which is exactly (15). $\square$

4. PROOF OF THEOREM 1

In this section, we give the proof of Theorem 1. We begin with rational surfaces.

Lemma 4.1. *If $X = \mathbb{P}^2$ or $X = \mathbb{F}_n$ with $n = 0$ or $n \geq 2$, then*

$$\mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+ = \mathcal{K}_X.$$

Proof. The projective plane and the Hirzebruch surfaces are compact smooth toric Kähler manifolds. By [Sha26, Theorem C], every Kähler class on a compact smooth toric manifold contains a torus-invariant PSC Kähler metric. Hence $\mathcal{K}_X^{\text{psc}} = \mathcal{K}_X$, which completes the proof. $\square$

We next turn to non-rational minimal ruled surfaces. We first treat the semistable case.

Lemma 4.2. *Let $X = \mathbb{P}(E) \rightarrow \Sigma_g$, where $g \geq 1$ and E is slope-semistable. Then*

$$(20) \quad \mathcal{K}_X^{\text{psc}} = \mathcal{T}_X^+ = \{2\pi(a\zeta + bF) : a > 0, \quad b > (g - 1)a\}.$$

Proof. Suppose first that E is polystable. By [ATF06, Theorem 2], $\mathbb{P}(E)$ admits a locally symmetric cscK metric in every Kähler class. If the class belongs to $\mathcal{T}_X^+$, the constant scalar curvature of this metric is positive. Thus $\mathcal{T}_X^+ \subseteq \mathcal{K}_X^{\text{psc}}$.

Fix a Kähler class $\alpha = 2\pi(a\zeta + bF) \in \mathcal{T}_X^+$ on X . Suppose next that E is semistable but not polystable. Then E is strictly semistable and admits the Jordan–Hölder filtration

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0, \quad \deg L = \deg Q = \frac{\deg E}{2}.$$

Scaling the extension class gives a holomorphic family $\mathcal{E} \rightarrow \Sigma_g \times \Delta$ such that

$$E_0 = L \oplus Q, \quad E_t \simeq E \quad \text{for } t \neq 0.$$

Set $p : \mathcal{X} = \mathbb{P}(\mathcal{E}) \rightarrow \Delta$ and $X_t = \mathbb{P}(E_t)$. The normalized tautological class and the fibre class define a class $\mathfrak{a} \in H^2(\mathcal{X}, \mathbb{R})$, admitting a smooth closed real $(1,1)$ -representative, such that $\mathfrak{a}|_{X_t} = \alpha_t := 2\pi(a\zeta_t + bF_t)$.

Since $E_0 = L \oplus Q$ is polystable, the central class α_0 contains a positive cscK metric. Choose a smooth closed real $(1,1)$ -form $\theta \in \mathfrak{a}$. By $\partial\bar{\partial}$ -lemma, there exists φ_0 on X_0 such that $\omega_0 = \theta|_{X_0} + \sqrt{-1}\partial\bar{\partial}\varphi_0$. After extending φ_0 smoothly to a neighborhood of X_0 , denote the extension by Φ . We then define $\omega_t = (\theta + \sqrt{-1}\partial\bar{\partial}\Phi)|_{X_t}$. For all sufficiently small $0 < |t| \ll 1$, ω_t is still Kähler and has positive scalar curvature on X_t . Since $E_t \simeq E$, the original class α contains a PSC Kähler metric. Finally, (7) with $\Delta(E) = 0$ gives (20). $\square$

It remains to determine the cone when E is unstable.

Lemma 4.3. *Let $X = \mathbb{P}(E) \rightarrow \Sigma_g$, where $g \geq 1$ and E is unstable. Then*

$$(21) \quad \mathcal{K}_X^{\text{psc}} = \left\{ 2\pi(a\zeta + bF) : a > 0, \quad b > \left(g - 1 + \frac{\Delta(E)}{2} \right) a \right\}.$$

Proof. Let $\alpha = 2\pi(a\zeta + bF)$ satisfy $a > 0$ and $b > \left(g - 1 + \frac{\Delta(E)}{2} \right) a$. Consider the Harder–Narasimhan filtration

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0, \quad \deg L - \deg Q = \Delta(E) > 0.$$

Scaling the extension class gives a holomorphic family $\mathcal{E} \rightarrow \Sigma_g \times \Delta$ with

$$E_0 = L \oplus Q, \quad E_t \simeq E \quad \text{for } t \neq 0.$$

After twisting by L^{-1} , the central ruled surface is

$$\mathbb{P}(E_0) \simeq \mathbb{P}(\mathcal{O}_{\Sigma_g} \oplus M), \quad M = Q \otimes L^{-1}, \quad \deg M = -\Delta(E).$$

Set $X_t = \mathbb{P}(E_t)$, and denote by ζ_t and F_t the normalized tautological and fibre classes on X_t . On the decomposable central fibre, in the notation of Appendix A, set $m = a$ and $\lambda = \frac{1}{\Delta(E)} \left(b - \frac{\Delta(E)}{2} a \right)$. Lemma A.1 therefore gives a PSC Kähler metric in the central class $2\pi(a\zeta_0 + bF_0)$. The same argument as in the proof of Lemma 4.2 gives a family of Kähler metrics with positive scalar curvature for all sufficiently small $t > 0$ on X_t . Since $E_t \simeq E$ for $t \neq 0$, the original class α contains a PSC Kähler metric. Combining with Proposition 3.2, we obtain (21). $\square$

We can now complete the proof of the classification for minimal surfaces.

Proof of Theorem 1. A minimal compact Kähler surface of Kodaira dimension $-\infty$ is either $\mathbb{P}^2$, a minimal Hirzebruch surface $\mathbb{F}_n$ with $n = 0$ or $n \geq 2$, or a geometrically ruled surface over a curve of positive genus; see [BHPVdV04, Chapter VI]. Lemma 4.1 proves the assertion (i). Now let $X = \mathbb{P}(E) \rightarrow \Sigma_g$ with $g \geq 1$. Lemma 4.2 computes the PSC cone when $\Delta(E) = 0$,

while Lemma 4.3 computes it when $\Delta(E) > 0$. Combining with (7), we prove the assertion (ii). $\square$

5. PROOF OF THEOREM 2

We first show the following statement.

Proposition 5.1. *Let Y be a smooth projective surface. Assume that there exist a $\mathbb{Q}$ -ample class A with $c_1(Y) \cdot A > 0$ and a normal ample test configuration $(\mathcal{Y}, \mathcal{A})$ for (Y, A) whose total space is smooth, with reduced central fibre, such that*

$$\mathcal{F}^{\text{NA}}(\mathcal{Y}, \mathcal{A}) < 0.$$

Then for every birational morphism $q : X \rightarrow Y$ from a smooth projective surface, $\mathcal{K}_X^{\text{psc}} \subsetneq \mathcal{T}_X^+$.

Proof. By [Bea96, Theorem II.11], every birational morphism between smooth projective surfaces factors as a finite sequence of blowups at closed points. Thus write

$$X = X_N \xrightarrow{\pi_N} X_{N-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = Y,$$

where $X_i = \text{Bl}_{p_i} X_{i-1}$. The points may be infinitely near.

We explain how to lift one blowup. Over $\mathbb{C}^*$, the test configuration is equivariantly trivial. The orbit of any point $p \in Y$ therefore defines a $\mathbb{C}^*$ -equivariant section of $\mathcal{Y}|_{\mathbb{C}^*} \rightarrow \mathbb{C}^*$. Properness extends this section uniquely across the origin. Denote its image by $\mathcal{Z}_p$. Since the family map composed with the section is the identity, the family is a submersion along $\mathcal{Z}_p$. Hence $\mathcal{Z}_p$ is a smooth codimension-two center contained in the smooth locus of every fibre.

Blow up this section: $\rho : \tilde{\mathcal{Y}} = \text{Bl}_{\mathcal{Z}_p} \mathcal{Y} \rightarrow \mathcal{Y}$, and let $\mathcal{D}$ be the exceptional divisor. For every sufficiently small rational $\varepsilon > 0$, $\tilde{\mathcal{A}}_\varepsilon = \rho^* \mathcal{A} - \varepsilon \mathcal{D}$ is relatively ample and defines a test configuration for $\pi^* A - \varepsilon D$ on $\text{Bl}_p Y$, where $\pi : \tilde{Y} = \text{Bl}_p Y \rightarrow Y$ is the blowup and $D \subset \tilde{Y}$ is its exceptional curve. The total space remains smooth and the central fibre remains reduced.

The non-Archimedean functional is given by intersection numbers on the compactified test configuration as in (14). For the above fixed blowup, these intersection numbers are polynomial in ε , while the general-fibre volume tends to A^2 . Consequently,

$$(22) \quad \lim_{\varepsilon \searrow 0} \mathcal{F}^{\text{NA}}(\tilde{\mathcal{Y}}, \tilde{\mathcal{A}}_\varepsilon) = \mathcal{F}^{\text{NA}}(\mathcal{Y}, \mathcal{A}).$$

Since the right-hand side is strictly negative, the lifted slope is negative for every sufficiently small $\varepsilon > 0$.

We now iterate this procedure. At the i -th stage, after the previous parameters have been fixed, the point $p_i \in X_{i-1}$ again defines an orbit section of the lifted test configuration. This remains true when p_i lies on a previous exceptional divisor. Choose successively $0 < \varepsilon_i \ll 1$ so that relative ampleness and strict negativity of the slope are preserved at each stage. We obtain a rational Kähler class A_N on X , defined recursively by

$$A_i = \pi_i^* A_{i-1} - \varepsilon_i D_i, \quad A_0 = A,$$

and a normal ample test configuration $(\mathcal{X}_N, \mathcal{A}_N)$ for X_N, A_N satisfying

$$(23) \quad \mathcal{F}^{\text{NA}}(\mathcal{X}_N, \mathcal{A}_N) < 0.$$

Moreover,

$$\begin{aligned} c_1(X_i) \cdot A_i &= (\pi_i^* c_1(X_{i-1}) - D_i) \cdot (\pi_i^* A_{i-1} - \varepsilon_i D_i) \\ &= c_1(X_{i-1}) \cdot A_{i-1} - \varepsilon_i. \end{aligned}$$

The parameters can therefore be chosen so that $c_1(X) \cdot A_N > 0$. Thus $2\pi A_N \in \mathcal{T}_X^+$.

If $2\pi A_N$ contained a PSC Kähler metric, Lemma 3.1 would force $\mathcal{F}^{\text{NA}} \geq \delta J^{\text{NA}} \geq 0$ for every normal test configuration. This contradicts (23). Hence

$$2\pi A_N \in \mathcal{T}_X^+ \setminus \mathcal{K}_X^{\text{psc}},$$

which proves the proposition. $\square$

We can now give the proof.

Proof of Theorem 2. Since X is Kähler and bimeromorphic to the projective surface Y , it is projective. Let C be the Harder–Narasimhan section of $Y = \mathbb{P}(E)$. Choose positive rational numbers a, b satisfying

$$(24) \quad \max \left\{ g-1, \frac{\Delta(E)}{2} \right\} a < b < \left(g-1 + \frac{\Delta(E)}{2} \right) a.$$

Then $A = a\zeta + bF$ is ample and satisfies $c_1(Y) \cdot A > 0$. Moreover,

$$A \cdot C = b - \frac{\Delta(E)}{2} a < (g-1)a.$$

Choose a rational number c such that

$$\frac{A \cdot C}{g-1} < c < a.$$

The deformation to the normal cone of C , with polarization $\overline{\mathcal{A}}_c$, is smooth, relatively ample, and has reduced central fibre. Formula (19) gives

$$\mathcal{F}^{\text{NA}}(c) = \frac{2c}{A^2} (A \cdot C - (g-1)c) < 0.$$

Proposition 5.1 now applies to the given morphism $q : X \rightarrow Y$, and proves the theorem. $\square$

APPENDIX A. THE MOMENTUM CONSTRUCTION ON DECOMPOSABLE RULED SURFACES

In the appendix, we give a specific construction of the PSC Kähler cone on decomposable minimal ruled surfaces by a method different from that used in the main text.

A.1. The S^1 -invariant PSC cone. Let $X = \mathbb{P}(\mathcal{O}_{\Sigma_g} \oplus L) \xrightarrow{\pi} \Sigma_g$ with $g \geq 1$ and $\deg L = -d < 0$. The splitting $\mathcal{O}_{\Sigma_g} \oplus L$ induces a natural fibrewise S^1 -action on X , obtained by scalar multiplication on the L -factor. We define the S^1 -invariant positive scalar curvature cone by

$$\mathcal{K}_{X,S^1}^{\text{psc}} := \{ \alpha \in \mathcal{K}_X : \exists \omega \in \alpha, \text{ } S^1\text{-invariant, s.t. } S(\omega) > 0 \}.$$

The purpose of this subsection is to compute this cone. In particular, any Kähler class on X can be written as $\alpha_{\lambda,m} := 2\pi(mC_0 + d(\lambda + m)F)$.

Choose a constant-scalar-curvature Kähler metric ω_Σ on Σ_g satisfying $[\omega_\Sigma] = -2\pi c_1(L)$. Then

$$\int_{\Sigma_g} \omega_\Sigma = 2\pi d, \quad \text{and} \quad S(\omega_\Sigma) = \frac{2-2g}{d} := s \leq 0,$$

thanks to the Gauss–Bonnet formula. We use the admissible momentum construction on decomposable ruled surfaces, in the spirit of Calabi ansatz [Cal82], following the Hwang–Singer momentum construction for circle-invariant Kähler metrics [HS02].

Let $L^\times$ denote the complement of the zero section in the total space of L . Choose a Hermitian metric h on L such that, on $L^\times$,

$$\sqrt{-1} \partial \bar{\partial} \log |v|_h^2 = \pi^* \omega_\Sigma.$$

Set $\tau := \log |v|_h^2$. Define the real one-form

$$\eta := \frac{\sqrt{-1}}{2} (\bar{\partial} - \partial) \tau.$$

Then

$$d\eta = \sqrt{-1} \partial \bar{\partial} \tau = \pi^* \omega_\Sigma.$$

On each fibre, if $v = \rho e^{\sqrt{-1}\theta}$ in a local trivialization, then $\eta = d\theta$. Thus η is the connection one-form induced by the Hermitian metric h on the principal S^1 -bundle of unit vectors in L , extended radially to $L^\times$.

We now choose an admissible radial potential. Let f be a smooth strictly convex function such that $f'(\mathbb{R}) = (0, m)$. Define the momentum coordinate $r = f'(\tau)$. Since f' is a diffeomorphism from $\mathbb{R}$ onto $(0, m)$, there is a smooth function ϕ on $(0, m)$ determined by

$$\phi(f'(\tau)) = f''(\tau).$$

We require that ϕ extends smoothly to $[0, m]$, such that $\phi(r) > 0$ for $0 < r < m$, and satisfies the standard endpoint conditions

$$(25) \quad \phi(0) = \phi(m) = 0, \quad \phi'(0) = 1, \quad \phi'(m) = -1.$$

Conversely, every smooth function ϕ satisfying these positivity and endpoint conditions arises from such a function f : indeed, one solves

$$\frac{dt}{dr} = \frac{1}{\phi(r)}$$

on $(0, m)$. The endpoint conditions (25) imply

$$t \rightarrow -\infty \quad \text{as } r \rightarrow 0, \quad t \rightarrow +\infty \quad \text{as } r \rightarrow m,$$

so $t : (0, m) \rightarrow \mathbb{R}$ is a diffeomorphism. Its inverse $r = r(t)$ then defines f , up to an additive constant, by $f'(t) = r(t)$. Then

$$f''(t) = \phi(r(t)) > 0.$$

On $L^\times$, define the admissible metric

$$\omega := \lambda \pi^* \omega_\Sigma + \sqrt{-1} \partial \bar{\partial} f(\tau).$$

Since $dr = f''(\tau) d\tau = \phi(r) d\tau$, we have

$$\begin{aligned} \sqrt{-1} \partial \bar{\partial} f(\tau) &= f'(\tau) \sqrt{-1} \partial \bar{\partial} \tau + f''(\tau) \sqrt{-1} \partial \tau \wedge \bar{\partial} \tau \\ &= r \pi^* \omega_\Sigma + dr \wedge \eta. \end{aligned}$$

Therefore $\omega = (\lambda + r) \pi^* \omega_\Sigma + dr \wedge \eta$. The form is Kähler on $L^\times$ precisely when $\lambda + r > 0$ and $\phi(r) > 0$. Since $0 < r < m$, the first condition follows from $\lambda > 0$. The endpoint conditions (25) guarantee that ω extends smoothly to X , and its Kähler class is

$$[\omega] = 2\pi(mC_0 + d(\lambda + m)F) = \alpha_{\lambda, m}.$$

Set $G(r) := (\lambda + r)\phi(r)$. Then the endpoint conditions (25) become

$$(26) \quad G(0) = G(m) = 0, \quad G'(0) = \lambda, \quad G'(m) = -(\lambda + m).$$

We next compute the scalar curvature. Since

$$(27) \quad \omega^2 = 2(\lambda + r) \pi^* \omega_\Sigma \wedge dr \wedge \eta,$$

the Ricci form is

$$(28) \quad \text{Ric}(\omega) = \pi^* \text{Ric}(\omega_\Sigma) - \sqrt{-1} \partial \bar{\partial} \log G.$$

For every smooth function $u = u(r)$, one has

$$(29) \quad \sqrt{-1} \partial \bar{\partial} u = (\phi u')' dr \wedge \eta + \phi u' \pi^* \omega_\Sigma,$$

where primes denote derivatives with respect to r . Applying this to $u = \log G$, we get

$$\phi(\log G)' = \frac{G'}{\lambda + r}.$$

Hence

$$(\lambda + r)(\phi(\log G)')' + \phi(\log G)' = G''.$$

Since $\text{Ric}(\omega_\Sigma) = s \omega_\Sigma$, equations (28) and (29) give

$$2 \text{Ric}(\omega) \wedge \omega = 2(s - G''(r)) \pi^* \omega_\Sigma \wedge dr \wedge \eta.$$

Therefore, we obtain

$$(30) \quad S(\omega) = \frac{2 \text{Ric}(\omega) \wedge \omega}{\omega^2} = \frac{s - G''(r)}{\lambda + r}.$$

Lemma A.1. *The class $\alpha_{\lambda, m} := 2\pi(mC_0 + d(\lambda + m)F)$ contains an admissible Kähler metric with positive scalar curvature if and only if $2\lambda + sm > 0$.*

Proof. Suppose that an admissible positive scalar-curvature metric exists. Then G satisfies the endpoint condition (26) and, by the scalar-curvature formula,

$$G''(r) < s \quad \text{for } 0 < r < m.$$

Equivalently, we may write $G'' = s - \mu$ for a smooth positive function μ on $(0, m)$. Indeed, Taylor's expansion with integral remainder gives

$$G(m) = G(0) + mG'(0) + \int_0^m (m - r)G''(r) dr.$$

Since $G(0) = G(m) = 0$ and $G'(0) = \lambda$, this becomes

$$\int_0^m (m-r)G''(r) dr = -m\lambda.$$

Substituting $G'' = s - \mu$, we obtain

$$\int_0^m (m-r)\mu(r) dr = \lambda m + \frac{s}{2}m^2.$$

The left-hand side is strictly positive, and hence $2\lambda + sm > 0$.

Conversely, assume $2\lambda + sm > 0$. Define

$$A := 2\lambda + (s+1)m, \quad B := m\left(\lambda + \frac{s}{2}m\right).$$

Then $B > 0$. Moreover,

$$A = (2\lambda + sm) + m > 0,$$

and

$$mA - B = m(2\lambda + (s+1)m) - m\left(\lambda + \frac{s}{2}m\right) = B + m^2 > 0.$$

Thus $B < mA$.

Choose a smooth positive function μ on $[0, m]$ such that

$$(31) \quad \int_0^m \mu(r) dr = A, \quad \int_0^m (m-r)\mu(r) dr = B.$$

This is possible since $B/A \in (0, m)$. In particular, one can choose a smooth positive probability density on $[0, m]$ whose barycenter with respect to the function $m-r$ is B/A , and multiply it by A .

Define G by

$$G'' = s - \mu, \quad G(0) = 0, \quad G'(0) = \lambda.$$

The first condition in (31) gives

$$G'(m) = G'(0) + \int_0^m G''(r) dr = \lambda + sm - A = -(\lambda + m).$$

The second condition in (31) gives

$$\begin{aligned} G(m) &= G(0) + mG'(0) + \int_0^m (m-r)G''(r) dr \\ &= \lambda m + \frac{s}{2}m^2 - B \\ &= 0. \end{aligned}$$

Therefore G satisfies the endpoint condition (26).

Since $G'' = s - \mu < 0$, the function G is strictly concave. Together with $G(0) = G(m) = 0$, this implies $G(r) > 0$ for $0 < r < m$. Define

$$\phi(r) := \frac{G(r)}{\lambda + r}.$$

Then $\phi > 0$ on $(0, m)$, $\phi \in C^\infty([0, m])$, and satisfies the endpoint condition (25).

Thus ϕ defines an admissible Kähler metric in $\alpha_{\lambda, m}$ with positive scalar curvature by (30), because $G'' < s$. $\square$

Proposition A.2. *The S^1 -invariant positive scalar curvature cone is*

$$\mathcal{K}_{X, S^1}^{\text{psc}} = \{\alpha_{\lambda, m} \in \mathcal{K}_X : 2\lambda + sm > 0\}.$$

Proof. The inclusion $\{\alpha_{\lambda,m} \in \mathcal{K}_X; 2\lambda + sm > 0\} \subseteq \mathcal{K}_{X,S^1}^{\text{psc}}$ follows directly from Lemma A.1, since admissible metrics are S^1 -invariant.

It remains to prove the reverse inclusion. Let $\omega \in \alpha_{\lambda,m}$ be an S^1 -invariant Kähler metric with $S(\omega) > 0$. Let ξ be the real holomorphic vector field generating the fibrewise S^1 -action. Since the holomorphic S^1 -action preserves ω and has a nonempty fixed locus, Frankel's theorem [Fra59] implies that the action is Hamiltonian. Thus there exists a moment map μ_ω with $\iota_\xi \omega = -d\mu_\omega$. The fixed locus of the S^1 -action on X has two connected components, namely the zero section D_0 and the infinity section D_∞ . Normalize the moment map by $\mu_\omega|_{D_0} = 0$. Since the generator has period 2π and

$$\int_F \omega = 2\pi m,$$

the difference of the moment map between the two fixed sections is m . Hence $\mu_\omega|_{D_\infty} = m$. With this normalization, the image of μ_ω is the interval $[0, m]$. Define

$$h_\omega := m - \mu_\omega.$$

Then $h_\omega \geq 0$ and $h_\omega \not\equiv 0$. Since $S(\omega) > 0$, it follows that

$$I(\omega) := \int_X h_\omega S(\omega) \omega^2 > 0.$$

We claim that $I(\omega)$ depends only on the Kähler class $\alpha_{\lambda,m}$, with the above normalization of the moment map. Indeed,

$$I(\omega) = \int_X h_\omega (S(\omega) - \overline{S}_{\alpha_{\lambda,m}}) \omega^2 + \overline{S}_{\alpha_{\lambda,m}} \int_X h_\omega \omega^2.$$

Since $dh_\omega = -d\mu_\omega = \iota_\xi \omega$, the function h_ω is a Hamiltonian function for $-\xi$, under our sign convention. Up to the normalization, the first term is the Futaki invariant of $-\xi$, which depends only on the Kähler class and the holomorphic vector field [FM95]. We next show that

$$I_0 = \int_X h_\omega \omega^2$$

also depends only on the Kähler class. Let $\omega_0, \omega_1 \in \alpha_{\lambda,m}$ be two S^1 -invariant Kähler forms and set

$$\omega_t := (1-t)\omega_0 + t\omega_1, \quad 0 \leq t \leq 1.$$

Then ω_t is an S^1 -invariant Kähler form in the same class. Let μ_t be its moment map. Because $\omega_1 - \omega_0$ is exact and S^1 -invariant, there is an S^1 -invariant real one-form γ such that

$$\dot{\omega}_t = \omega_1 - \omega_0 = d\gamma.$$

Differentiating $\iota_\xi \omega_t = -d\mu_t$ with respect to t , we obtain

$$\iota_\xi d\gamma = -d\dot{\mu}_t.$$

Since γ is S^1 -invariant, Cartan's formula gives

$$\iota_\xi d\gamma = \mathcal{L}_\xi \gamma - d(\iota_\xi \gamma) = -d(\iota_\xi \gamma).$$

It follows that

$$d(\dot{\mu}_t - \iota_\xi \gamma) = 0.$$

Hence $\dot{\mu}_t - \iota_\xi \gamma$ is constant. On D_0 , the normalization gives $\dot{\mu}_t = 0$, while $\xi = 0$ because D_0 is fixed. Therefore, the constant vanishes and $\dot{\mu}_t = \iota_\xi \gamma$. We now differentiate the first moment of the moment map,

$$\begin{aligned} \frac{d}{dt} \int_X \mu_t \omega_t^2 &= \int_X \dot{\mu}_t \omega_t^2 + 2 \int_X \mu_t d\gamma \wedge \omega_t \\ &= \int_X (\iota_\xi \gamma) \omega_t^2 - 2 \int_X d\mu_t \wedge \gamma \wedge \omega_t \\ &= \int_X (\iota_\xi \gamma) \omega_t^2 + 2 \int_X \iota_\xi \omega_t \wedge \gamma \wedge \omega_t \\ &= 0 \end{aligned}$$

Thus, I_0 is independent of the chosen S^1 -invariant representative $\omega \in \alpha_{\lambda, m}$. Since

$$I_0 = \int_X h_\omega \omega^2 = m \int_X \omega^2 - \int_X \mu_\omega \omega^2,$$

and $\int_X \omega^2$ depends only on the Kähler class, the second term is also class-invariant. Consequently $I(\omega)$ depends only on $\alpha_{\lambda, m}$, and we may denote it by $I(\alpha_{\lambda, m})$. We can compute $I(\alpha_{\lambda, m})$ by using any admissible metric in the same class. For an admissible metric ω , the normalized moment map μ_ω is precisely the momentum coordinate r . Hence $h_\omega = m - r$. Using (30) and (27), we obtain, for some positive constant C ,

$$I(\alpha_{\lambda, m}) = C \int_0^m (m - r)(s - G''(r)) dr.$$

Using the endpoint conditions (26), we have

$$\begin{aligned} \int_0^m (m - r)G''(r) dr &= [(m - r)G'(r)]_0^m + \int_0^m G'(r) dr \\ &= -mG'(0) + G(m) - G(0) \\ &= -m\lambda. \end{aligned}$$

Therefore

$$I(\alpha_{\lambda, m}) = C \left(\frac{s}{2} m^2 + \lambda m \right) = \frac{Cm}{2} (2\lambda + sm) > 0.$$

Since $C > 0$ and $m > 0$, we obtain $2\lambda + sm > 0$. This proves the reverse inclusion. $\square$

A.2. Symmetrization and the full PSC cone. We first record the symmetrization statement needed below.

Lemma A.3. *Let a compact Lie group G act holomorphically on a compact Kähler manifold Y , and let $\beta \in \mathcal{K}_Y$ be a G -invariant Kähler class. Assume that $\beta \in \mathcal{K}_Y^{\text{psc}}$. Then β contains a G -invariant Kähler metric of positive scalar curvature.*

Proof. Choose a representative Kähler metric $\omega_0 \in \beta$. Since β is G -invariant, the averaged form

$$\omega := \int_G g^* \omega_0 dg$$

is a G -invariant Kähler metric in the same class β , where dg denotes the normalized Haar measure on G .

Since $\beta \in \mathcal{K}_Y^{\text{psc}}$, there exists a Kähler metric of positive scalar curvature in β . Set $\Omega = \overline{S}_\beta \omega^n$. By [Sha26, Theorem A], the prescribed scalar-curvature measure equation

$$(32) \quad S(\omega_\varphi) \omega_\varphi^n = \Omega, \quad \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \in \beta,$$

admits a smooth solution φ . Moreover, for fixed Ω , the solution is unique up to addition of constants to φ .

For every $g \in G$, since $g^*\omega = \omega$ and $g^*\Omega = \Omega$, the potential $g^*\varphi$ gives another solution of (32). By uniqueness, $g^*\omega_\varphi = \omega_\varphi$. Thus ω_φ is G -invariant. Finally,

$$S(\omega_\varphi) = \frac{\Omega}{\omega_\varphi^n} > 0.$$

Hence ω_φ is a G -invariant Kähler metric of positive scalar curvature in β . $\square$

We now give the concrete formula for the full PSC Kähler cone.

Proposition A.4. *Let $X = \mathbb{P}(\mathcal{O}_{\Sigma_g} \oplus L) \rightarrow \Sigma_g$ with $\deg L = -d < 0$, and let $\alpha_{\lambda,m} := 2\pi(mC_0 + d(\lambda + m)F)$ be a Kähler class. Then*

$$\mathcal{K}_X^{\text{psc}} = \left\{ \alpha_{\lambda,m} \in \mathcal{K}_X : 2\lambda - \frac{2m(g-1)}{d} > 0 \right\}.$$

In particular, if $g \geq 2$, then $\mathcal{K}_X^{\text{psc}} \subsetneq \mathcal{T}_X^+$.

Proof. By Proposition A.2,

$$\mathcal{K}_{X,S^1}^{\text{psc}} = \{ \alpha_{\lambda,m} \in \mathcal{K}_X : 2\lambda + sm > 0 \} \subseteq \mathcal{K}_X^{\text{psc}}.$$

Conversely, suppose that $\alpha_{\lambda,m} \in \mathcal{K}_X^{\text{psc}}$. The natural fibrewise S^1 -action on X is connected, hence acts trivially on cohomology. Therefore the class $\alpha_{\lambda,m}$ is S^1 -invariant. Applying Lemma A.3 to this S^1 -action, we obtain an S^1 -invariant Kähler metric of positive scalar curvature in the class $\alpha_{\lambda,m}$. Hence $\alpha_{\lambda,m} \in \mathcal{K}_{X,S^1}^{\text{psc}}$. Thus

$$\mathcal{K}_X^{\text{psc}} = \mathcal{K}_{X,S^1}^{\text{psc}} = \{ \alpha_{\lambda,m} \in \mathcal{K}_X : \lambda > 0, m > 0, 2\lambda + sm > 0 \}.$$

It remains to compute $\mathcal{T}_X^+$. Since $c_1(X) = 2C_0 + (d + 2 - 2g)F$ and

$$C_0^2 = -d, \quad C_0 \cdot F = 1, \quad F^2 = 0,$$

we have

$$\begin{aligned} c_1(X) \cdot \alpha_{\lambda,m} &= 2\pi(2C_0 + (d + 2 - 2g)F) \cdot (mC_0 + d(\lambda + m)F) \\ &= 2\pi(-2dm + 2d(\lambda + m) + m(d + 2 - 2g)) \\ &= 2\pi d \left(2\lambda + \left(1 + \frac{2 - 2g}{d} \right) m \right) \\ &= 2\pi d(2\lambda + (1 + s)m). \end{aligned}$$

Since $d > 0$, this gives

$$\mathcal{T}_X^+ = \{ \alpha_{\lambda,m} \in \mathcal{K}_X : 2\lambda + (1 + s)m > 0 \}.$$

Therefore $\mathcal{K}_X^{\text{psc}} \subseteq \mathcal{T}_X^+$. In particular, the inclusion is strict when $g \geq 2$. Assume now that $g \geq 2$, so that $s < 0$. Fix $m > 0$ and set $\lambda = -sm/2 > 0$. Then $2\lambda + sm = 0$, whereas $2\lambda + (1 + s)m = m > 0$. Hence $\alpha_{\lambda,m} \in \mathcal{T}_X^+ \setminus \mathcal{K}_X^{\text{psc}}$. Consequently, $\mathcal{K}_X^{\text{psc}} \subsetneq \mathcal{T}_X^+$. $\square$

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