Computing the canonical bundle of a blow-up via short exact sequences

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Abstract

This note presents my personal study on computing the canonical bundle of a blow-up over some submanifolds on a compact complex manifold, using short exact sequences. Note that only short exact sequences can save my life.

1 Introduction

Let X^n be a compact complex manifold, and let $Y_i \subset X$ be some submanifold of X with codimension m_i. The blow-up of X over Y_i is a n-dimensional complex manifold $\tilde{X}$ together with a holomorphic map $\pi : \tilde{X} \to X$, such that:

- $E_i := \pi^{-1}(Y_i)$ is a hypersurface in $\tilde{X}$, which is called the exceptional divisor, and the restriction map $\pi|_{E_i} : E_i \to Y_i$ is a holomorphic fiber bundle isomorphic to the projectivized normal bundle with each fiber isomorphic to $\mathbb{CP}^{m_i-1}$;
- $\pi : \tilde{X} \setminus (\cup_i E_i) \to X \setminus (\cup_i Y_i)$ is a biholomorphism.

Then, the canonical line bundle of $\tilde{X}$ is given by

$$K_{\tilde{X}} = \pi^* K_X \otimes_{\text{for all } i} (m_i - 1)\mathcal{O}(E_i). \tag{1}$$

2 Computing the formula

Without loss of generality, we only compute the formula when Y_i is a point p in X. The general case for blow-up over arbitrary submanifolds can be obtained in the same way.

In this case, we have the exceptional divisor $E \cong \mathbb{CP}^{n-1}$. Consider the short exact sequence for the relative cotangent sheaf:

$$0 \to \pi^*\Omega^1_X \to \Omega^1_{\tilde{X}} \to \Omega^1_{\tilde{X}/X} \to 0.$$

Taking the determinant, we have

$$\begin{aligned} K_{\tilde{X}} &= \det\left(\pi^*\Omega^1_{\tilde{X}}\right) \otimes \det\left(\Omega^1_{\tilde{X}/X}\right) \\ &= \pi^* K_X \otimes \det\left(\Omega^1_{\tilde{X}/X}\right). \end{aligned}$$

Our goal is to see what is $\det\left(\Omega^1_{\tilde{X}/X}\right)$. We restrict the the relative cotangent sheaf on E, and note that $\pi^*\Omega^1_X|_E = \Omega^1_X|_p \cong \mathcal{O}_E$, we obtain

$$0 \to \mathcal{O}_E \to \Omega^1_{\tilde{X}}\big|_E \to \Omega^1_{\tilde{X}/X}\big|_E \to 0.$$

On the other hand, we know the conormal bundle sequence,

$$0 \to \mathcal{N}^*_{E/\tilde{X}} \to \Omega^1_{\tilde{X}}\big|_E \to \Omega^1_E \to 0.$$

This gives

$$\begin{aligned}
\det\left(\Omega^1_{\tilde{X}/X}\right)\Big|_E &= \det\left(\mathcal{N}^*_{\tilde{X}/E}\right) \otimes \det\left(\Omega^1_E\right) \\
&= \mathcal{O}_E(1) \otimes \mathcal{O}_E(-n) \\
&= (n-1)\mathcal{O}_E(-1) \\
&= (n-1)\mathcal{O}(E)|_E,
\end{aligned}$$

where we use the fact that $K_{\mathbb{CP}^n} \cong \mathcal{O}_{\mathbb{CP}^n}(-n-1)$ and $\mathcal{N}_D \cong \mathcal{O}(D)|_D$ for a divisor. Recall that the Picard group for the blow-up is given by

$$\operatorname{Pic}(\tilde{X}) = \pi^* \operatorname{Pic}(X) \oplus \mathbb{Z}[E],$$

which implies any line bundle $\tilde{L}$ over $\tilde{X}$ must have the form $\tilde{L} \cong \pi^* L \otimes k\mathcal{O}(E)$, for some $L \in \operatorname{Pic}(X)$ and $k \in \mathbb{Z}$. Since $\det\left(\Omega^1_{\tilde{X}/X}\right)$ is a line bundle support on E which is trivial when push-forward over X, we have

$$\det\left(\Omega^1_{\tilde{X}/X}\right) \cong k\mathcal{O}(E).$$

We obtain $k = n-1$ by our computation when restricting to E. Thus we have

$$K_{\tilde{X}} = \pi^* K_X \otimes (n-1)\mathcal{O}(E).$$

The formula of the canonical bundle for blow-up over several points on X can be obtained by blow-up again at a point over $\tilde{X}$ and repeatedly applying this formula several times.