

EVEN-SYSTOLIC INEQUALITIES ON KÄHLER FIBRATIONS

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ABSTRACT. We prove sharp even-dimensional systolic inequalities for positive scalar curvature Kähler metrics on holomorphic Kähler fibrations over bases with non-positive Ricci curvature.

1. INTRODUCTION

Systolic geometry studies how topology forces the existence of cycles of small volume on a Riemannian manifold. Since the foundational works of Berger [2, 1, 3] and Gromov [7], systolic inequalities have become an important theme in global Riemannian geometry. Under positive scalar curvature assumptions, however, sharp systolic inequalities are usually difficult to obtain in dimensions at least four. In dimension three, the interaction between scalar curvature and area-minimizing surfaces led to sharp results of Bray–Brendle–Neves [4], building on the Schoen–Yau theory of stable minimal surfaces [13]. Stern later gave a new proof using a level-set method. There has also been recent progress on higher-dimensional and quantitative variants; see for instance [16, 12, 11]. More recently, stable 2-systolic inequalities under positive scalar curvature have been studied widely by Stryker [15], Cecchini–Hirsch–Zeidler [5], and Kumar [9].

The Kähler category provides additional information. Complex subvarieties are calibrated by the Kähler form, so their volumes are controlled by cohomological intersection numbers. Moreover, starting from any rank $r + 1$ holomorphic vector bundle over a compact Kähler manifold, its projectivization naturally produces a fibration with fibres \mathbb{P}^r , and these fibres contain the linear subspaces \mathbb{P}^q for $1 \leq q \leq r$, which serve as canonical candidates for the $2q$ -systoles. This makes even-dimensional systoles particularly natural in the Kähler setting.

In our previous work [14], we proved a sharp 2-systole inequality on compact Kähler surfaces with positive scalar curvature. In particular, for non-rational ruled surfaces one obtains the optimal estimate

$$\min_X S(\omega) \cdot \text{sys}_2(\omega) \leq 4\pi,$$

with equality only in the locally product model $\mathbb{P}^1 \times E$, where E is an elliptic curve.

1.1. Even-dimensional systolic inequality. The purpose of this note is to present a higher-dimensional analogue of the 2-systole inequality on non-rational ruled surfaces. Let $E \rightarrow Y$ be a holomorphic vector bundle of rank $r + 1$, and set $X_0 = \mathbb{P}(E)$. Suppose that X is obtained from X_0 by a finite sequence of blow-ups along smooth centers whose images are contained in proper analytic subsets of Y . Then the induced map

$$\pi : X \longrightarrow Y$$

is still a holomorphic submersion with fibre \mathbb{P}^r outside a proper analytic subset of Y . Thus the theorem below applies not only to projective bundles, but also to these blow-ups, and gives a sharp $2q$ -systole estimate for every $1 \leq q \leq r$.

Theorem A. *Let $\pi : X^{m+r} \rightarrow Y^m$ be a holomorphic fibration between compact Kähler manifolds. Assume that, outside a proper analytic subset $\Delta \subset Y$, the map π is a holomorphic submersion and every regular fibre is biholomorphic to \mathbb{P}^r . Suppose that X admits a Kähler metric ω with positive scalar curvature and Y admits a Kähler metric η with non-positive Ricci curvature. Then, for every $1 \leq q \leq r$,*

$$\boxed{\min_X S(\omega) \cdot \text{sys}_{2q}(\omega)^{1/q} \leq 2\pi r(r+1).}$$

Moreover, for a fixed q , equality if and only if

$$(\tilde{X}, \tilde{\omega}) = (\mathbb{P}^r, \omega_{FS}) \times (\tilde{Y}, \omega_Y), \quad \text{Ric}(\omega_Y) = 0,$$

with $\text{sys}_{2q}(\omega)$ realized by a fibrewise linear $\mathbb{P}^q \subset \mathbb{P}^r$.

1.2. Two consequences for the 2-systole. When $m = r = q = 1$, the fibre is \mathbb{P}^1 and Theorem A gives

$$\min_X S(\omega) \cdot \text{sys}_2(\omega) \leq 4\pi,$$

recovering the 2-systole inequality on non-rational ruled surfaces in [14, Theorem 1.2(3)].

We record two further consequences of the same method for the 2-systole. They are not literally covered by Theorem A: in the first one the regular fibre is not assumed in advance to be \mathbb{P}^1 , while in the second one the regular fibre may be a Picard number one Fano manifold different from projective space.

Theorem B. *Let $\pi : X^{n+1} \rightarrow Y^n$ be a holomorphic fibration between compact Kähler manifolds. Suppose that X admits a Kähler metric ω with positive scalar curvature and that Y admits a Kähler metric η with non-positive Ricci curvature. Then*

$$\boxed{\min_X S(\omega) \cdot \text{sys}_2(\omega) \leq 4\pi.}$$

Moreover, equality holds if and only if

$$(\tilde{X}, \tilde{\omega}) = (\mathbb{P}^1, \omega_{FS}) \times (\tilde{Y}, \omega_Y), \quad \text{Ric}(\omega_Y) = 0,$$

with $\text{sys}_2(\omega)$ realized by a \mathbb{P}^1 -fibre.

Proof. Indeed, the same fibrewise scalar inequality as Lemma 3.1 gives

$$\min_X S(\omega) \int_Y \text{Vol}_\omega(F_y) \eta^n \leq \int_Y \left(\int_{F_y} S(\omega_y) \omega_y \right) \eta^n.$$

By Gauss–Bonnet,

$$\int_{F_y} S(\omega_y) \omega_y = 2\pi \chi(F_y) \leq 4\pi.$$

Since $S(\omega) > 0$, the right-hand side must be positive, so the regular fibre has positive Euler characteristic and hence is biholomorphic to \mathbb{P}^1 . The inequality then follows from

$$\text{sys}_2(\omega) \leq \text{Vol}_\omega(F_y).$$

The equality case is the same as in Theorem A. In particular, the 2-systole is realized by a \mathbb{P}^1 -fibre. \square

Theorem C. *Let $\pi : X^{m+r} \rightarrow Y^m$ be a holomorphic fibration between compact Kähler manifolds. Assume that, outside a proper analytic subset $\Delta \subset Y$, the map π is a holomorphic submersion and every regular fibre is biholomorphic to a fixed Fano manifold F^r with $\rho(F) = 1$. Suppose that*

X admits a Kähler metric ω with positive scalar curvature and that Y admits a Kähler metric η with non-positive Ricci curvature. Then

$$\boxed{\min_X S(\omega) \cdot \text{sys}_2(\omega) \leq 2\pi r(r+1).}$$

Moreover, equality holds if and only if $F \cong \mathbb{P}^r$ and

$$(\tilde{X}, \tilde{\omega}) = (\mathbb{P}^r, \omega_{FS}) \times (\tilde{Y}, \omega_Y), \quad \text{Ric}(\omega_Y) = 0,$$

with $\text{sys}_2(\omega)$ realized by a projective line $\mathbb{P}^1 \subset \mathbb{P}^r$.

Proof. Let h be the primitive ample generator of $N^1(F)$. Since F is Fano and $\rho(F) = 1$, we may write

$$c_1(F) = I_F h,$$

where I_F is the Fano index. Moreover, since F is Fano with $\rho(F) = 1$, one has $H^2(F, \mathcal{O}_F) = 0$ and $H^{1,1}(F; \mathbb{R})$ is one-dimensional. Therefore the restriction of the Kähler class to a regular fibre is of the form

$$[\omega_y] = ah$$

for some $a > 0$. The same fibrewise scalar inequality as Lemma 3.1 then gives

$$\min_X S(\omega) \cdot a \leq 2\pi r I_F.$$

By Mori's bend-and-break technique[10], there exists a rational curve $C \subset F$ such that

$$-K_F \cdot C \leq r + 1.$$

Since $-K_F = I_F h$, we have

$$h \cdot C \leq \frac{r+1}{I_F}.$$

Thus

$$\text{sys}_2(\omega) \leq \text{Vol}_\omega(C) = ah \cdot C \leq a \frac{r+1}{I_F}.$$

Combining the two inequalities gives

$$\min_X S(\omega) \cdot \text{sys}_2(\omega) \leq 2\pi r(r+1).$$

If equality holds, then equality holds in the rational curve bound, so the pseudoindex of F is $r+1$. By the theorem of Cho–Miyaoka–Shepherd-Barron [6], $F \cong \mathbb{P}^r$. The remaining rigidity follows as in Theorem A. \square

Remark 1.1. *Examples include smooth quadrics Q^r in dimension at least three, complex Grassmannians $\text{Gr}(k, N)$, parametrizing k -dimensional linear subspaces of \mathbb{C}^N , and many smooth Fano hypersurfaces or complete intersections with Picard number one. The corresponding fibrations need not be products. For instance, if $E \rightarrow Y$ is a holomorphic vector bundle of rank N , then the relative Grassmannian $\text{Gr}(k, E) \rightarrow Y$ has fibre $\text{Gr}(k, N)$, a Picard number one Fano manifold. Choosing E not projectively trivial gives a non-product example. Similarly, a holomorphic vector bundle of rank $r+2$ equipped with a fibrewise non-degenerate quadratic form gives a relative quadric fibration, whose fibres are smooth quadrics Q^r .*

2. PRELIMINARIES

2.1. Notations. Let (Z^d, ω) be a compact Kähler manifold of complex dimension d . In local coordinates (z^1, \dots, z^d) , the Kähler metric $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. We denote by

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{i\bar{j}})$$

the Ricci form of ω . The first Chern class

$$2\pi c_1(Z) = [\text{Ric}(\omega)].$$

The Chern scalar curvature is

$$S(\omega) := \text{tr}_\omega \text{Ric}(\omega).$$

Equivalently,

$$(1) \quad S(\omega) \omega^d = d \text{Ric}(\omega) \wedge \omega^{d-1}.$$

Consequently,

$$(2) \quad \int_Z S(\omega) \omega^d = 2\pi d \int_Z c_1(Z) \wedge \omega^{d-1}.$$

When $d = 1$, this is the Gauss–Bonnet formula. In particular, if α is a real $(1, 1)$ -form on Z , then

$$(\text{tr}_\omega \alpha) \omega^d = d \alpha \wedge \omega^{d-1}.$$

We shall use this identity repeatedly without further comment.

2.2. Set-up. Let

$$\pi : X^{m+r} \longrightarrow Y^m$$

be a holomorphic fibration between compact Kähler manifolds. Assume that there exists a proper analytic subset $\Delta \subset Y$ such that, if

$$Y^\circ := Y \setminus \Delta, \quad X^\circ := \pi^{-1}(Y^\circ),$$

then

$$\pi^\circ := \pi|_{X^\circ} : X^\circ \longrightarrow Y^\circ$$

is a holomorphic submersion and every fibre of π° is biholomorphic to \mathbb{P}^r . We write

$$\dim_{\mathbb{C}} X = n, \quad \dim_{\mathbb{C}} Y = m, \quad r := n - m.$$

For $y \in Y^\circ$, we denote the regular fibre by

$$F_y := \pi^{-1}(y).$$

Then F_y is biholomorphic to \mathbb{P}^r .

Let ω be a Kähler metric on X , and let η be a Kähler metric on Y . Along a regular fibre F_y , set

$$\omega_y := \omega|_{F_y}.$$

Then we have $[\omega_y] = ah$ on $F_y \cong \mathbb{P}^r$ for some $a > 0$, where $h = c_1(\mathcal{O}_{\mathbb{P}^r}(1))$ is the hyperplane class. Since the regular fibres form a smooth family over Y° , the number a is independent of $y \in Y^\circ$. In particular,

$$(3) \quad \int_{F_y} \omega_y^r = a^r, \quad \int_{\mathbb{P}^q \subset F_y} \omega_y^q = a^q \quad (1 \leq q \leq r).$$

Since π° is a holomorphic submersion, along every regular fibre F_y there is a short exact sequence

$$0 \longrightarrow T^{1,0} F_y \longrightarrow T^{1,0} X|_{F_y} \longrightarrow T^{1,0} X|_{F_y} / T^{1,0} F_y := N_{F_y/X} \longrightarrow 0.$$

Taking the dual sequence and then taking the determinant gives

$$K_X|_{F_y} = K_{F_y} \otimes (\det N_{F_y/X})^*.$$

Since, $N_{F_y/X}$ is trivial, we have

$$c_1(X)|_{F_y} = c_1(F_y).$$

Also we have,

$$(4) \quad \int_{F_y} S(\omega_y) \omega_y^r = 2\pi r \int_{F_y} c_1(F_y) \wedge \omega_y^{r-1}.$$

Since $F_y \cong \mathbb{P}^r$ and $c_1(F_y) = (r+1)h$, this gives

$$(5) \quad \int_{F_y} S(\omega_y) \omega_y^r = 2\pi r(r+1)a^{r-1}.$$

Finally, let $V := \ker \partial\pi^\circ \subset T^{1,0}X|_{X^\circ}$ be the vertical tangent bundle, and let $N := V^\perp$ be its ω -orthogonal complement. Thus $T^{1,0}X|_{X^\circ} = V \oplus N$ as an orthogonal splitting. If e_1, \dots, e_r is an ω -unitary frame of V , and e_{r+1}, \dots, e_n is an ω -unitary frame of N , we write

$$\mathrm{tr}_V \mathrm{Ric}(\omega) := \sum_{i=1}^r \mathrm{Ric}(\omega)(e_i, \bar{e}_i), \quad \mathrm{tr}_N \mathrm{Ric}(\omega) := \sum_{\alpha=1}^m \mathrm{Ric}(\omega)(e_{r+\alpha}, \bar{e}_{r+\alpha}).$$

Then pointwise along each regular fibre,

$$(6) \quad S(\omega) = \mathrm{tr}_V \mathrm{Ric}(\omega) + \mathrm{tr}_N \mathrm{Ric}(\omega).$$

The vertical trace is related to the scalar curvature of the fibre after integration. Indeed,

$$(7) \quad \int_{F_y} \mathrm{tr}_V \mathrm{Ric}(\omega) \omega_y^r = \int_{F_y} S(\omega_y) \omega_y^r.$$

To see this, note that

$$[\mathrm{Ric}(\omega)|_{F_y}] = 2\pi c_1(X)|_{F_y} = 2\pi c_1(F_y) = [\mathrm{Ric}(\omega_y)].$$

Hence $\mathrm{Ric}(\omega)|_{F_y} - \mathrm{Ric}(\omega_y)$ is $\sqrt{-1}\partial\bar{\partial}$ -exact, thanks to the $\partial\bar{\partial}$ -lemma. Therefore, Stokes' theorem gives

$$\int_{F_y} \mathrm{Ric}(\omega)|_{F_y} \wedge \omega_y^{r-1} = \int_{F_y} \mathrm{Ric}(\omega_y) \wedge \omega_y^{r-1}.$$

which yields (7).

2.3. The Bochner formula for complex Jacobian. The complex differential of π is a holomorphic bundle morphism

$$\partial\pi : T^{1,0}X \longrightarrow \pi^*T^{1,0}Y.$$

Taking its top exterior power gives

$$\Lambda^m \partial\pi : \Lambda^m T^{1,0}X \longrightarrow \pi^* \Lambda^m T^{1,0}Y.$$

Equivalently,

$$s_\pi := \Lambda^m \partial\pi \in \Gamma(\Lambda^m(T^{1,0}X)^* \otimes \pi^* \Lambda^m T^{1,0}Y)$$

is a holomorphic section. We define the complex Jacobian of π by

$$J_\pi := |s_\pi|_{\omega, \eta}^2.$$

Let (\mathcal{E}, h) be a Hermitian holomorphic vector bundle over a Kähler manifold (Z, ω) , and let $\Theta(\mathcal{E}, h)$ be its Chern curvature. If s is a holomorphic section of \mathcal{E} , then

$$(8) \quad \Delta_\omega |s|_h^2 = |\nabla s|^2 - \langle \sqrt{-1} \mathrm{tr}_\omega \Theta(\mathcal{E}, h) s, s \rangle_h.$$

This is the standard Chern-Bochner identity for holomorphic sections; see, for example, [8, eq. (4)]. We use the convention

$$\Delta_\omega = g^{i\bar{j}} \nabla_i \nabla_{\bar{j}}.$$

Lemma 2.1. *At every point $x \in X^\circ$, choose an ω -unitary frame*

$$e_1, \dots, e_r, e_{r+1}, \dots, e_n$$

such that e_1, \dots, e_r span $V = \ker \partial\pi^\circ$, and choose an η -unitary frame

$$\varepsilon_1, \dots, \varepsilon_m$$

at $\pi(x)$ such that

$$\partial\pi(e_{r+\alpha}) = \lambda_\alpha \varepsilon_\alpha, \quad \lambda_\alpha > 0, \quad \alpha = 1, \dots, m.$$

Then

$$(9) \quad \Delta_\omega J_\pi = |\nabla s_\pi|^2 + J_\pi \sum_{\alpha=1}^m \text{Ric}(\omega)(e_{r+\alpha}, \overline{e_{r+\alpha}}) - J_\pi \sum_{\alpha=1}^m \lambda_\alpha^2 \text{Ric}(\eta)(\varepsilon_\alpha, \overline{\varepsilon_\alpha}).$$

Proof. Apply (8) to the holomorphic section

$$s_\pi \in \Gamma(\mathcal{E}), \quad \mathcal{E} = \Lambda^m(T^{1,0}X)^* \otimes \pi^* \Lambda^m T^{1,0}Y.$$

Let $\theta^1, \dots, \theta^n$ be the coframe dual to e_1, \dots, e_n . At x ,

$$s_\pi = \lambda_1 \cdots \lambda_m \theta^{r+1} \wedge \cdots \wedge \theta^n \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_m.$$

Hence

$$J_\pi = |s_\pi|^2 = \lambda_1^2 \cdots \lambda_m^2.$$

The curvature of \mathcal{E} is the sum of the curvatures of the two tensor factors. First, the curvature of $(T^{1,0}X)^*$ is the negative transpose of the curvature of $T^{1,0}X$. Therefore, one has

$$\langle \sqrt{-1} \text{tr}_\omega \Theta(\Lambda^m(T^{1,0}X)^*) s_\pi, s_\pi \rangle = -J_\pi \sum_{\alpha=1}^m \text{Ric}(\omega)(e_{r+\alpha}, \overline{e_{r+\alpha}}).$$

Second,

$$\Lambda^m T^{1,0}Y = K_Y^{-1},$$

and the Chern curvature of K_Y^{-1} is represented by the Ricci form ρ_η . Pulling back to X and contracting with ω , we get

$$\langle \sqrt{-1} \text{tr}_\omega \Theta(\pi^* \Lambda^m T^{1,0}Y) s_\pi, s_\pi \rangle = J_\pi \sum_{i=1}^n \text{Ric}(\eta)(\partial\pi(e_i), \overline{\partial\pi(e_i)}).$$

Since $\partial\pi(e_i) = 0$ for $1 \leq i \leq r$, and

$$\partial\pi(e_{r+\alpha}) = \lambda_\alpha \varepsilon_\alpha,$$

this becomes

$$J_\pi \sum_{\alpha=1}^m \lambda_\alpha^2 \text{Ric}(\eta)(\varepsilon_\alpha, \overline{\varepsilon_\alpha}).$$

Substituting the two curvature contributions into (8) gives (9). \square

2.4. **The co-area formula.** We record the co-area formula for the smooth submersion

$$\pi^\circ : (X^\circ, \omega) \longrightarrow (Y^\circ, \eta).$$

Since $Y \setminus Y^\circ$ is a proper analytic subset, it has measure zero. Thus, in our normalization, the formula may be written as follows.

Lemma 2.2. *For every smooth function $\Phi \in C^\infty(X)$,*

$$(10) \quad \int_X J_\pi \Phi \omega^n = \frac{n!}{r!m!} \int_Y \left(\int_{F_y} \Phi \omega_y^r \right) \eta^m.$$

3. PROOF OF THEOREM A

We keep the notation from Section 2.

Lemma 3.1. *Assume that*

$$\text{Ric}(\eta) \leq 0$$

as a Hermitian form on Y . Then

$$(11) \quad \int_Y \left(\int_{F_y} S(\omega) \omega_y^r \right) \eta^m \leq \int_Y \left(\int_{F_y} S(\omega_y) \omega_y^r \right) \eta^m.$$

Moreover, equality in (11) holds if and only if $\nabla s_\pi \equiv 0$ and $\text{Ric}(\eta) \equiv 0$.

Proof. By the Bochner formula (9), at every point of X° one has

$$\Delta_\omega J_\pi = |\nabla s_\pi|^2 + J_\pi \text{tr}_N \text{Ric}(\omega) - J_\pi \sum_{\alpha=1}^m \lambda_\alpha^2 \text{Ric}(\eta)(\varepsilon_\alpha, \bar{\varepsilon}_\alpha).$$

Since $\text{Ric}(\eta) \leq 0$, the last term is non-negative. Hence

$$\Delta_\omega J_\pi \geq |\nabla s_\pi|^2 + J_\pi \text{tr}_N \text{Ric}(\omega).$$

Since $X \setminus X^\circ$ has measure zero, integrating over X with respect to the Kähler volume form ω^n , and using compactness of X , gives

$$(12) \quad 0 = \int_X \Delta_\omega J_\pi \omega^n \geq \int_X J_\pi \text{tr}_N \text{Ric}(\omega) \omega^n.$$

By the co-area formula (10), this is equivalent to

$$\int_Y \left(\int_{F_y} \text{tr}_N \text{Ric}(\omega) \omega_y^r \right) \eta^m \leq 0.$$

Integrating (6) over the regular fibre and using (7), we obtain

$$\int_{F_y} S(\omega) \omega_y^r = \int_{F_y} S(\omega_y) \omega_y^r + \int_{F_y} \text{tr}_N \text{Ric}(\omega) \omega_y^r.$$

Integrating this identity over Y and combining with (12) give (11).

It remains to record the equality case. Equality in (11) is equivalent to

$$\int_X J_\pi \text{tr}_N \text{Ric}(\omega) \omega^n = 0.$$

The Bochner formula (9) then tells us

$$0 = - \int_X |\nabla s_\pi|^2 \omega^n + \int_X J_\pi \sum_{\alpha=1}^m \lambda_\alpha^2 \text{Ric}(\eta)(\varepsilon_\alpha, \bar{\varepsilon}_\alpha) \omega^n.$$

Note that both terms are non-positive. Therefore equality holds if and only if both terms vanish:

$$\nabla s_\pi \equiv 0$$

and

$$\sum_{\alpha=1}^m \lambda_\alpha^2 \operatorname{Ric}(\eta)(\varepsilon_\alpha, \overline{\varepsilon_\alpha}) \equiv 0.$$

Since $\operatorname{Ric}(\eta) \leq 0$, the last identity forces

$$\operatorname{Ric}(\eta)(\varepsilon_\alpha, \overline{\varepsilon_\alpha}) = 0 \quad \text{for every } \alpha.$$

As $\varepsilon_1, \dots, \varepsilon_m$ form a unitary frame at an arbitrary point of Y° , and Y° is dense in Y , this is equivalent to $\operatorname{Ric}(\eta) \equiv 0$. \square

We now can give the proof of our main theorem.

Proof of Theorem A. Applying Lemma 3.1, we get

$$\int_Y \left(\int_{F_y} S(\omega) \omega_y^r \right) \eta^m \leq \int_Y \left(\int_{F_y} S(\omega_y) \omega_y^r \right) \eta^m.$$

The left-hand side is bounded from below by

$$\min_X S(\omega) \int_Y \left(\int_{F_y} \omega_y^r \right) \eta^m = \min_X S(\omega) \cdot a^r \int_Y \eta^m.$$

By (5), the right-hand side is

$$2\pi r(r+1)a^{r-1} \int_Y \eta^m.$$

Since $a > 0$ and $\int_Y \eta^m > 0$, we obtain

$$(13) \quad \min_X S(\omega) \cdot a \leq 2\pi r(r+1).$$

Now fix $1 \leq q \leq r$. Let

$$\mathbb{P}^q \subset F_y \cong \mathbb{P}^r$$

be a linear subspace in a regular fibre. It defines a non-trivial $2q$ -homology class in X . Since $\omega_y = ah$ for some $a > 0$ and h is the hyperplane class, we have

$$\operatorname{Vol}_\omega(\mathbb{P}^q) = \int_{\mathbb{P}^q} \omega^q = a^q.$$

Therefore, any $2q$ -systole in X satisfies

$$\operatorname{sys}_{2q}(\omega) \leq a^q.$$

Combining this with (13), we obtain

$$\min_X S(\omega) \cdot \operatorname{sys}_{2q}(\omega)^{1/q} \leq \min_X S(\omega) \cdot a \leq 2\pi r(r+1),$$

which proves the desired inequality.

We finally discuss the equality case. Since $S(\omega) > 0$, equality in the systolic inequality forces equality at each step of the proof. Hence

$$\operatorname{sys}_{2q}(\omega) = a^q, \quad S(\omega) \equiv \min_X S(\omega),$$

and equality in Lemma 3.1 gives

$$\operatorname{Ric}(\eta) \equiv 0, \quad \nabla s_\pi \equiv 0.$$

Moreover, since $J_\pi > 0$ on X° , the parallel section s_π has constant non-zero norm. Hence $J_\pi > 0$ on X , and π has no critical points. Thus the $2q$ -systole is realized by a fibrewise linear $\mathbb{P}^q \subset \mathbb{P}^r$.

The condition $\nabla s_\pi = 0$ has the usual splitting consequence. Indeed,

$$V = \ker \partial\pi = \{Z \in T^{1,0}X : \iota_Z s_\pi = 0\},$$

and differentiating $\iota_Z s_\pi = 0$ shows that V is parallel. Hence $N = V^\perp$ is parallel as well, and the de Rham splitting theorem gives a Kähler-isometric splitting of the universal cover:

$$(\tilde{X}, \tilde{\omega}) \cong (F, \omega_F) \times (\tilde{Y}, \omega_Y), \quad F \cong \mathbb{P}^r.$$

Since the base factor is Ricci-flat and $S(\omega)$ is constant, the fibre metric is cscK in the class ah . Therefore ω_F is the scaled Fubini–Study metric, up to automorphism. This is the claimed rigidity model, and the converse is immediate. \square

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