THE KÄHLER-RICCI SOLITON ON BOUNDED PSEUDOCONVEX DOMAINS

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ABSTRACT. In this paper, we study the Kähler-Ricci soliton on bounded pseudoconvex domains in $\mathbb{C}^n$ with C^2-boundary. Under certain assumptions, we prove that such solitons reduce to Kähler-Einstein metrics. Building on Huang and Xiao's resolution of Cheng's conjecture, we further establish an analogous rigidity result for the Bergman Kähler–Ricci soliton. Several model domains are provided to illustrate our results.

1. INTRODUCTION

In Kähler geometry, a central problem is to find the "best" metric in a given Kähler class. The Kähler-Einstein metric is widely regarded as a canonical candidate for this problem. Despite this, not every Kähler manifold admits a Kähler-Einstein metric. To broaden this framework, Kähler-Ricci solitons generalize Kähler-Einstein metric that consists of a Kähler metric g, a real holomorphic vector field X and a constant $\lambda \in \mathbb{R}$, satisfying the equation

$$\mathrm{Ric}(g) + \mathcal{L}_X g = \lambda g.$$

Moreover, if there exist potential functions $f \in C^\infty(M)$, such that $X = \frac{1}{2}\nabla f$, we call g a gradient Kähler-Ricci soliton. A Kähler-Ricci soliton g is said to be trivial if X is a Killing vector field, in which case g reduces to a Kähler-Einstein metric with Einstein constant λ. In [Ham88], Hamilton first showed that all compact gradient steady and expanding solitons are Einstein metrics. Combined with Perelman's work [Per02], which shows that all compact solitons are gradient, it follows that there are no non-trivial compact Kähler–Ricci solitons if the first Chern class of the Kähler manifold is non-positive. On compact Fano manifolds (that is, Kähler manifolds with positive first Chern class), if there exists a non-trivial Kähler-Ricci soliton g with a non-trivial real holomorphic vector field X, then the Futaki invariant for $X^{1,0}$,

$$F(X^{1,0}) = \int_M |X^{1,0}|^2 \omega_g^n$$

is strictly positive. Consequently, the existence of a non-trivial Kähler-Ricci soliton obstructs the existence of a Kähler-Einstein metric [Fut83].

Perelman's breakthrough revealed that all steady and expanding solitons must be non-compact. The compactness assumption is crucial for applying powerful tools such as the $\partial\overline{\partial}$-lemma and the maximum principle. This limitation makes the study of canonical metrics on complete K"ahler manifolds significantly less developed compared to their compact counterparts. Bounded pseudoconvex domains in $\mathbb{C}^n$ provide an ideal setting for addressing this gap. These

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domains retain essential geometric features of non-compact Kähler manifolds while being simpler to analyze.

Notably, Mok, and Yau [MY83] showed that a bounded domain carries a Kähler-Einstein metric if and only if it is pseudoconvex. Despite some progress of significant research for non-compact gradient Kähler-Ricci soliton ([FIK03, Cao96, Cao97]), the existence of Kähler-Ricci solitons remains unexplored on pseudoconvex domains.

In this paper, we study the case of a bounded domain Ω equipped with a complete Kähler metric with C^2-boundary. It is known that a bounded domain in a complex manifold with C^1-boundary admitting a Kähler metric is pseudoconvex [Ohs80]. Our main result is the following:

Theorem 1.1. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^2-boundary, and let g be a complete Kähler metric on Ω of C^1-bounded geometry. Suppose there exists a compact subset $K \subset\subset \Omega$ such that

$$-C_1 g \geq \mathrm{Ric}(g) \geq -C_2 g \quad \text{in} \quad \Omega \setminus K,$$

for $C_2 \geq C_1 > 0$. If g is a Kähler-Ricci soliton with a real holomorphic vector field X such that the dual 1-form of X with respect to g is closed, then g is Kähler-Einstein.

As an important application, based on Huang-Xiao's proof [HX21] for Cheng's conjecture, we have the following analogous statement:

Corollary 1.2. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞-boundary and let g_B be the Bergman metric. If g_B is a Kähler-Ricci soliton, then Ω is biholomorphic to the ball.

On the compact Kähler manifold with negative first Chern class, i.e. admits a Kähler-Einstein metric with negative Ricci curvature, it is well known that every Kähler–Ricci soliton is trivial. In contrast, for complete manifolds such as bounded pseudoconvex domains with C^2-boundary, which always admit a Kähler–Einstein metric—the situation is less clear. An interesting question naturally arises from these considerations:

Question 1.3. Does there exist any non-trivial Kähler-Ricci soliton on bounded pseudoconvex domains with C^2-boundary?

Our main result states that if the Ricci curvature of a complete Kähler metric is asymptotically negative, then the metric cannot admit any soliton structure unless it is Kähler-Einstein. In fact, if a real holomorphic vector field is the gradient of some potential function, then its dual 1-form is closed. However, the converse is not true due to potential topological obstructions.

Nonetheless, since not all complete Kähler metrics satisfy this asymptotic Ricci curvature condition (for example, see [KY96] on the boundary behavior of the Bergman metric on pseudoconvex domains), one may ask: *Can we prove that every Kähler–Ricci soliton on bounded*

pseudoconvex domains with C^2-boundary is trivial, or can we construct an example that violates the assumptions of our main theorem? It is worth noting that many domains equipped with complete Kähler metrics satisfy our assumptions. In Sections 4–6, we provide detailed explanations and several examples as applications.

2. PRELIMINARIES

2.1. **Notations.** Let (M^n, g) be an n-dimensional complete Kähler manifold. In local coordinates $(z^1, ..., z^n)$, the Kähler metric can be expressed as

$$g = g_{i\bar{j}} dz^i d\bar{z}^j,$$

and we will use the Einstein summation convention. Denote the matrix of metric components by $\left(g_{i\bar{j}}\right)$ with the inverse $\left(g^{i\bar{j}}\right)$ for $1 \leq i, j \leq n$, satisfying $g^{k\bar{l}} g_{k\bar{j}} = \delta^l_j$. For any two tensors A, B of type (m, n), the Hermitian inner product of A, B with respect to g is defined by

$$g(A, \bar{B}) := g^{i_1\bar{j}_1} \cdots g^{i_n\bar{j}_n} g_{k_1\bar{l}_1} \cdots g_{k_m\bar{l}_m} A^{k_1\cdots k_m}_{i_1\cdots i_n} \overline{B^{l_1\cdots l_m}_{j_1\cdots j_n}},$$

and the norm of A is defined by $|A|^2 := g(A, \bar{A})$.

The Ricci curvature $\mathrm{Ric}(g)$ is defined by

$$\mathrm{Ric}(g) = R_{i\bar{j}} dz^i d\bar{z}^j \quad \text{where} \quad R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\left(\det(g_{k\bar{l}})\right).$$

And, the scalar curvature $R(g)$ is defined by the trace of the Ricci curvature with respect to g

$$R(g) = \mathrm{tr}_g\left(\mathrm{Ric}(g)\right) = g^{i\bar{j}} R_{i\bar{j}},$$

where ∂_i and $\partial_{\bar{j}}$ are short notations for $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \bar{z}^j}$ respectively.

Let ∇ be the associated Levi-Civita connection of the Kähler metric g. Covariant derivatives are denoted by

$$\nabla_i = \nabla_{\partial_i} \quad \text{and} \quad \nabla_{\bar{j}} = \nabla_{\partial_{\bar{j}}}.$$

It is remarkable that the covariant derivative $\nabla_i \nabla_{\bar{j}}$ coincides with the partial derivative $\partial_i \partial_{\bar{j}}$ when applied to C^2 functions, since mixed type Christoffel symbols $\Gamma^{\cdot}_{i\bar{j}}$ vanish. For any C^∞ function $f : M \to \mathbb{R}$, the gradient vector field of f is defined by

$$\nabla f := g^{i\bar{j}} \partial_i f \cdot \partial_{\bar{j}} + g^{i\bar{j}} \partial_{\bar{j}} f \cdot \partial_i.$$

Let $\nabla^2 f := \left(\partial_i \partial_{\bar{j}} f\right)$ be the complex Hessian of f, then the Laplacian is defined by the trace of $\nabla^2 f$ with respect to g, that is

$$\Delta f := \mathrm{tr}_g\left(\nabla^2 f\right) = g^{i\bar{j}} \partial_i \partial_{\bar{j}} f.$$

2.2. **Kähler-Einstein metric and Kähler-Ricci soliton.** We say a Kähler metric is Kähler-Einstein if the Ricci curvature is proportional to the metric, that is

$$\mathrm{Ric}(g) = \lambda g, \tag{2.1}$$

for some $\lambda \in \mathbb{R}$. By rescaling the metric, we can assume that $\lambda = 1,\ 0$ or -1.

A vector field X is said to be real if $\bar{X} = X$, where $\bar{X}$ is the complex conjugate of X. Recall that a real holomorphic vector field X is a real vector field such that its $(1,0)$-part, that is $X^{1,0} := X - \sqrt{-1}JX$ is holomorphic for the fixed complex structure J. We say a Kähler metric g is a Kähler-Ricci soliton if there is an associated real holomorphic vector field X and a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(g) + \mathcal{L}_X g = \lambda g, \tag{2.2}$$

where $\mathcal{L}_X g$ is the Lie derivative of the Kähler metric g along X. Suppose that there exists a C^∞ real-valued function f such that $X = \frac{1}{2}\nabla f$, then we say g is a gradient Kähler-Ricci soliton and f is the potential function. The assumption of ∇f being a real holomorphic vector field is equivalent to $\nabla_i \nabla_j f = 0$ for all $1 \leq i, j \leq n$. The Kähler-Ricci soliton is said to be shrinking if $\lambda > 0$, steady if $\lambda = 0$, and expanding if $\lambda < 0$. Note that if the Lie derivative term vanishes, the soliton equation (2.2) reduces to the Kähler-Einstein equation (2.1).

2.3. **Bounded geometry and Omori-Yau's maximum principle.** To establish our main result, we first introduce the concept of (quasi-)bounded geometry. Recall that the injectivity radius at a point $x \in M$ is the maximum radius r of the ball B_r in the tangent space $T_x M$ for which the exponential map $\exp_x : B_r \to \exp_x(B_r) \subset M$ is a diffeomorphism. The injectivity radius of M is the infimum of the injectivity radius at all points in M.

Definition 2.1. Let (M, g) be a complete Kähler manifold and let $k \geq 0$ be an integer. We say (M, g) has C^k-quasi-bounded geometry if for each non-negative integer $l \leq k$, there exists a constant $C_l > 0$ such that

$$\sup_M |\nabla^l \mathrm{Rm}| \leq C_l, \tag{2.3}$$

where $\mathrm{Rm} = \{R_{i\bar{j}k\bar{l}}\}$ is the Riemann curvature tensor of g and ∇^l is the covariant derivative of order l. Moreover, if (M, g) has a positive injectivity radius, then we say (M, g) has C^k-bounded geometry.

Next, let us introduce Omori-Yau's generalized maximum principle on non-compact manifolds which serves as a crucial tool in our approach.

Proposition 2.2 ([Omo67, Yau75]). Let (M, g) be a complete Kähler manifold. Assume that g has bounded sectional curvature, then for any function $u \in C^2(M)$ with $\sup_M u < +\infty$, there exists a sequence of points $\{z_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$(1)\ \lim_{k\to\infty} u(z_k) = \sup_M u,\ (2)\ \lim_{k\to\infty} |\nabla u(z_k)| = 0,\ (3)\ \lim_{k\to\infty} \nabla^2 u(z_k) \leq 0,$$

where the last inequality holds in the sense of matrices.

Remark 2.3. If we replace the assumption of bounded sectional curvature in the generalized maximum principle with bounded Ricci curvature, then (3) is replaced by

$$(3') \lim_{k\to\infty} \Delta u(z_k) \leq 0.$$

3. Proof of Theorem 1.1

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^2-boundary and let g be a complete Kähler metric on Ω. Suppose that g is a Kähler-Ricci soliton satisfying

$$\mathrm{Ric}(g) + \mathcal{L}_X g = \lambda g,$$

for a real holomorphic vector field X and a constant $\lambda \in \mathbb{R}$.

Proposition 3.1. If the Ricci curvature is bounded from below, that is $\mathrm{Ric}(g) \geq Cg$ for some constant C, then $\lambda < 0$.

Proof. We first claim that $C < 0$. Indeed, if $C \geq 0$, then when $C > 0$, Myers's theorem implies that (Ω, g) is compact, which is a contradiction. Moreover, complete manifolds with nonnegative Ricci curvature do not admit positive nonconstant harmonic functions [SY10, Corollary 3.1], so the case $C = 0$ is also excluded for bounded domains in $\mathbb{C}^n$. Hence, $C < 0$, and there exists a point $z \in \Omega$ such that $\mathrm{Ric}(g) = -\varepsilon g$ at z for some $\varepsilon > 0$ small enough. This implies that the scalar curvature satisfies $R(z) < 0$, which would contradict the sharp lower bound estimates for the scalar curvature of Ricci solitons [Cho23, Theorem 2.14] if $\lambda \geq 0$. □

Since our assumptions ensure that the Ricci curvature is negatively pinched near the boundary, it follows that the Ricci curvature is bounded from below globally. Without loss of generality, we may assume that $\lambda = -1$. We now recall the following equation for the scalar curvature R when $\lambda = -1$.

Lemma 3.2 ([AMR16], Proposition 8.3)**.** Let $S := R + n$, then

$$\Delta S - \langle X, \nabla S\rangle - S + |\mathrm{Ric}(g) + g|^2 = 0. \tag{3.1}$$

Observe that the scalar curvature attains its minimum only on the boundary, thanks to the sharp lower bound estimates for Ricci solitons. By applying the generalized maximum principle to (3.1), we can deduce the asymptotic behavior of the Kähler-Ricci soliton near boundary points where the scalar curvature reaches its minimum.

Proposition 3.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^2-boundary, and let g be a complete Kähler-Ricci soliton associated with a real holomorphic vector field X such that $|\mathrm{Ric}(g)| \leq K$ for some constant $K > 0$. Suppose $R(p) = -n$ for some $p \in \partial\Omega$, meaning that for any sequence $\{p_m\}$ converging to p we have $R(p_m) \to -n$. If

$$|X|(z) \leq |\nabla R|^{-1}(z) \cdot d(z, \partial\Omega)$$

for all $z \in \Omega$ where d denotes the Euclidean distance, then there exists a sequence $p_k \to p$ such that

$$\lim_{k\to\infty} |\operatorname{Ric}(g) + g|(p_k) = 0.$$

Proof. For any $p \in \partial\Omega$ such that $R(p) = -n$, we may choose a sequence $\{p_k\}$ provided by the generalized maximum principle. Then it follows from the equation (3.1),

$$0 \leq \lim_{k\to\infty} \Delta S(p_k) = \lim_{k\to\infty} \left(\langle \nabla S, X\rangle(p_k) + S(p_k) - |\operatorname{Ric}(g) + g|^2(p_k)\right).$$

Since $|X|(p_k) \leq |\nabla R|^{-1}(p_k) \cdot d(p_k, \partial\Omega)$, this implies that

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \Delta S(p_k) \\ &\leq \lim_{k\to\infty} d(p_k, \partial\Omega) + \lim_{k\to\infty} S(p_k) - \lim_{k\to\infty} |\operatorname{Ric}(g) + g|^2(p_k) \\ &\leq -\lim_{k\to\infty} |\operatorname{Ric}(g) + g|^2(p_k) \\ &\leq 0. \end{aligned}$$

Thus, we have

$$\lim_{k\to\infty} |\operatorname{Ric}(g) + g|(p_k) = 0,$$

which is the conclusion. □

First, we see the following Bochner formula for real holomorphic vector fields.

Lemma 3.4. Assume that g is a complete Kähler metric and X is a real holomorphic vector field, then we have

$$\Delta|X|^2 = |\nabla X|^2 - \operatorname{Ric}(X, X). \tag{3.2}$$

Proof. We compute this formula in local coordinates. The assumption of X being real holomorphic is equivalent to $\nabla_i X_j = 0$ for any i, j, and we use the cancellation notation $\cancel{\nabla_i X_j}$ to indicate when this property is applied. Let $\nabla^i = g^{i\bar{j}}\nabla_{\bar{j}}$, then we have

$$\begin{aligned} \Delta|X|^2 &= \nabla^i\nabla_i \left(X^k \cdot X_k\right) \\ &= \nabla^i \left(\nabla_i X^k \cdot X_k + \cancel{X^k \cdot \nabla_i X_k}\right) \\ &= \nabla^i X_k \cdot \nabla_i X^k + X_k \cdot \nabla^i\nabla_i X^k. \end{aligned}$$

Denote the first term by $|\nabla X|^2$ and commute ∇^i with ∇_i in the second term, we obtain

$$\begin{aligned} \Delta|X|^2 &= |\nabla X|^2 + X_k \cdot \left(-R_l{}^k \cdot X^l + \cancel{\nabla_i\nabla^i X^k}\right) \\ &= |\nabla X|^2 - R_{l\bar{k}} X^l X^{\bar{k}} \\ &= |\nabla X|^2 - \operatorname{Ric}(X, X), \end{aligned}$$

which completes the proof. □

Remark 3.5. On compact Kähler manifolds with negative first Chern class, as a direct consequence of the equation (3.2), there is no non-trivial holomorphic vector field.

Assume that g is a complete Kähler-Ricci soliton with C^1-bounded geometry. Denote $X^\flat = (X^\flat)^{1,0} + (X^\flat)^{0,1}$ the dual 1-form of the real holomorphic vector field X where $(X^\flat)^{0,1} := g(X^{1,0}, \cdot)$. If $X^\flat$ is closed, then for any complex vector field Y, Z, we have

$$0 = dX^\flat(Y, Z) = (\nabla_Y X^\flat)(Z) - (\nabla_Z X^\flat)(Y).$$

Next, we show that the real holomorphic vector field X is unique up to addition by a constant. Otherwise, suppose that X and $\tilde{X}$ are two real holomorphic vector fields for g with the same soliton constant whose dual 1-forms are closed. Setting $Y = X - \tilde{X}$, then for any $1 \leq i, j \leq n$, we deduce:

$$\begin{cases} \nabla_j Y_i = 0; \\ \nabla_{\bar{j}} Y_i = 0, \end{cases}$$

where the first equation follows from the real holomorphicity of Y, and the second equation follows from the soliton equations of X and $\tilde{X}$, respectively. Applying $g^{k\bar{j}}\nabla_k$ to the second equation, we have

$$0 = g^{k\bar{j}}\nabla_k\nabla_{\bar{j}} Y_i = g^{k\bar{j}}\left(\nabla_{\bar{j}}\nabla_k Y_i - R_{k\bar{j}i\bar{l}}Y^{\bar{l}}\right) = -R_{i\bar{l}}Y^{\bar{l}},$$

which implies $\mathrm{Ric}(Y, Y) = 0$. Since $\mathrm{Ric}(g)$ is negatively pinched outside a compact subset $K \subset\subset \Omega$, it follows that in $\Omega \setminus K$,

$$0 = \mathrm{Ric}(Y, Y) \leq -C|Y|^2,$$

for some constant $C > 0$. Hence, $Y = 0$ in $\Omega \setminus K$, and by the real analyticity of Y, we conclude that $Y = 0$ throughout Ω.

We now proceed to the proof of the main theorem.

Proof of Theorem 1.1. Consider the soliton equation in local coordinates,

$$R_{i\bar{j}} + \nabla_i X_{\bar{j}} + \nabla_{\bar{j}} X_i = -g_{i\bar{j}}. \tag{3.3}$$

Since $X^\flat$ is closed, we have

$$\nabla_i X_{\bar{j}} = \nabla_{\bar{j}} X_i.$$

Applying $g^{k\bar{j}}\nabla_k$ to both sides of (3.3), we obtain

$$g^{k\bar{j}}\nabla_k R_{i\bar{j}} + 2g^{k\bar{j}}\nabla_k\nabla_{\bar{j}} X_i = 0.$$

By using contracted Bianchi identity and commuting ∇_k with $\nabla_{\bar{j}}$, we deduce that

$$\begin{aligned} 0 =& g^{k\bar{j}}\nabla_k R_{i\bar{j}} + 2g^{k\bar{j}}\nabla_k\nabla_{\bar{j}} X_i \\ =& \nabla_i R + 2g^{k\bar{j}}\left(\cancel{\nabla_{\bar{j}}\nabla_k X_i} - R_{k\bar{j}i\bar{l}}X^{\bar{l}}\right), \end{aligned}$$

Therefore,

$$\nabla_i R = 2R_{i\bar{l}}X^{\bar{l}},$$

or, in a coordinate-free notation

$$g(\nabla R, \cdot) = 2\,\mathrm{Ric}(X, \cdot). \tag{3.4}$$

It is clear that $|X|$ is bounded on the compact subset $K \subset\subset \Omega$. So on $\Omega \setminus K$, we have $-C_1 g \leq \mathrm{Ric} \leq -C_2 g$ for some constant $C_1, C_2 > 0$. Then it follows from (3.4),

$$g(\nabla R, X) = 2\,\mathrm{Ric}(X, X) \leq -2C_2|X|^2,$$

which implies

$$|X|^2 \leq -\frac{1}{2C_2} g(\nabla R, X) \leq \frac{1}{2C_2}|X| \cdot |\nabla R|.$$

Thus, we have

$$\sup_\Omega |X| \leq \max\left\{\max_K |X|, \frac{\sup_\Omega |\nabla R|}{2C_2}\right\} < +\infty,$$

thanks to g has C^1-bounded geometry.

Substitute X into the soliton equation, we obtain

$$\mathrm{Ric}(X, X) + \mathcal{L}_X g(X, X) = -|X|^2.$$

Observe that $\mathcal{L}_X g(X, X) = \nabla_X |X|^2$, we then have

$$\mathrm{Ric}(X, X) = -\nabla_X |X|^2 - |X|^2.$$

Replacing the Ricci curvature in (3.2) with the above expression, we get

$$\Delta|X|^2 = |\nabla X|^2 + \nabla_X |X|^2 + |X|^2 \geq \nabla_X |X|^2 + |X|^2.$$

Since $\sup_\Omega |X| < +\infty$, by the generalized maximum principle, there exists a sequence $\{z_k\}$ such that

$$0 \geq \lim_{k\to\infty} \Delta_g |X|^2(z_k) \geq \lim_{k\to\infty} \nabla_X |X|^2(z_k) + \lim_{k\to\infty} |X|^2(z_k) = \sup_\Omega |X|^2 \geq 0,$$

which forces $|X| = 0$. Therefore, g is Kähler-Einstein. □

Remark 3.6. Note that the assumption that $X^\flat$ is closed does not imply that X is the gradient of a potential function. In the case of a gradient Kähler-Ricci soliton, one can apply the generalized maximum principle to the PDE

$$\Delta f - |\nabla f|^2 - f = 0,$$

for the potential function f, to deduce that $f = 0$, using the boundedness of f.

4. Two Kähler metrics on strictly pseudoconvex domains

4.1. **The Bergman metric.** Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ be the space of holomorphic functions in $L^2(\Omega)$. Clearly, $A^2(\Omega)$ is a Hilbert space. The Bergman kernel $K(z)$ on Ω is a real analytic function given by

$$K(z) = \sum_{j=1}^{\infty} |\varphi_j(z)|^2, \qquad \forall z \in \Omega,$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $A^2(\Omega)$ with respect to the L^2 inner product. Since the Bergman kernel is positive and independent of the choice of any orthonormal basis [Kra01] on bounded domains, we can define an invariant metric on Ω called the Bergman metric by

$$g_B := g_{i\bar{j}} dz^i d\bar{z}^j \quad \text{with} \quad g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log K.$$

The Bergman metric is a complete, real analytic Kähler metric, with its real analyticity inherited from the Bergman kernel. On strictly pseudoconvex domains with C^∞ boundary, the sectional curvature of the Bergman metric is asymptotic to a negative constant, and the metric has C^∞-bounded geometry [GKK11]. Moreover, due to the boundary behavior of the Ricci curvature of g_B [KY96, Corollary 2], there exists a compact subset $K \subset\subset \Omega$ such that $\mathrm{Ric}(g_B)$ is bounded and strictly negative outside K.

Let $G := \det\left(g_{i\bar{j}}\right)$ denote the determinant of the Bergman metric. The Bergman invariant function $B(z) := G(z)/K(z)$, introduced by Bergman in [BS51], is invariant under any biholomorphic map between two domains. A notable result by Diederich [Die70] asserts that as one approaches the boundary of a strictly pseudoconvex domain, $B(z)$ tends to $(n+1)^n \pi^n / n!$. Furthermore, it was shown in [FW97] that if $B(z)$ is constant throughout Ω, then the Bergman metric is Kähler-Einstein.

Proposition 4.1. Let Ω be a bounded strictly pseudoconvex domain with C^∞-boundary. Then the Bergman metric is a Kähler-Ricci soliton if and only if there exists $f \in C^\infty(\Omega)$ up to the addition of a pluriharmonic function such that $B = e^f$ on Ω.

Proof. The "if" part is straightforward. Assuming $B = e^f$ and applying $\sqrt{-1}\partial\bar{\partial}$ to both sides immediately yields the soliton equation.

We now prove the "only if" part. Let ω_B be the Kähler form for the Bergman metric. Assume there is a real holomorphic vector field, such that

$$\mathrm{Ric}(\omega_B) + \mathcal{L}_X \omega_B = -\omega_B.$$

Since $\mathcal{L}_X \omega_B$ is real and closed $(1,1)$-form, by the local $\partial\bar{\partial}$-lemma (see, e.g., [Mor07, Propsition 2.8]), there is an open subset $U \subset \Omega$ and a real function $f \in C^\infty(U)$ such that $\mathcal{L}_X \omega_B|_U = \sqrt{-1}\partial\bar{\partial} f$. Thus, on U we have

$$\mathrm{Ric}(\omega_B) + \sqrt{-1}\partial\bar{\partial} f = -\omega_B,$$

and additionally,

$$\sqrt{-1}\partial\bar{\partial}\left(\log B - f\right) = 0 \quad \text{on} \quad U.$$

Hence, there exists a pluriharmonic function h such that $B = e^{f+h} := e^{\tilde{f}}$ on U where $\tilde{f} := f + h$. Since both B and $\tilde{f}$ are real-analytic, the equality $B = e^{\tilde{f}}$ extends to the whole Ω uniquely, which completes the proof.

In fact,

$$\mathrm{Ric}(\omega_B) + \sqrt{-1}\partial\bar{\partial}\tilde{f} = -\omega_B$$

is analytic on U, which must also hold on Ω. This shows that the Bergman metric is a gradient Kähler-Ricci soliton. □

It was conjectured by S.-Y. Cheng [Che79] that if the Bergman metric of a bounded strictly pseudoconvex domain with C^∞-boundary is Kähler–Einstein, then the domain is biholomorphic to the ball. In [HX21], Huang and Xiao provided an affirmative answer to Cheng's conjecture.

Theorem 4.2 ([HX21], Theorem 1.1)**.** The Bergman metric of a bounded strictly pseudoconvex domain Ω with C^∞-boundary is Kähler-Einstein if and only if the domain is biholomorphic to the ball.

By combining Theorem 4.2 with Theorem 1.1, we immediately deduce the following:

Corollary 4.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^∞-boundary and let g_B be the Bergman metric. If g_B is a Kähler-Ricci soliton, then Ω is biholomorphic to the ball.

4.2. **The complete Kähler metric given by a defining function.** For the remaining part of this section, we assume that Ω admits a C^{k+2}, $k \geq 5$, defining function ρ such that $\rho = 0$, $d\rho \neq 0$ on $\partial\Omega$, $\left(\rho_{i\bar{j}}\right) > 0$ on $\overline{\Omega}$ and $\Omega = \{\rho < 0\}$, where $\rho_{i\bar{j}} := \partial_i\partial_{\bar{j}}\rho$. Let $\phi = -\log(-\rho)$, we can define another complete Kähler metric on Ω of C^{k-2}-bounded geometry introduced in [CY80], that is

$$g_\rho := g_{i\bar{j}}dz^i d\bar{z}^j \quad \text{where} \quad g_{i\bar{j}} := \partial_i\partial_{\bar{j}}\phi.$$

A direct computation gives

$$g_{i\bar{j}} = \frac{\rho_{i\bar{j}}}{-\rho} + \frac{\rho_i\rho_{\bar{j}}}{\rho^2}, \qquad g^{i\bar{j}} = (-\rho)\left(\rho^{i\bar{j}} + \frac{\rho^i\rho^{\bar{j}}}{\rho - |d\rho|^2}\right)$$

and

$$\begin{aligned}\det\left(g_{i\bar{j}}\right) &= \left(-\frac{1}{\rho}\right)^n \det\left(\rho_{i\bar{j}} - \frac{\rho_i\rho_{\bar{j}}}{\rho}\right) \\ &= \left(\frac{1}{\rho}\right)^{n+1} \det\left(\rho_{i\bar{j}}\right)\left(-\rho + |d\rho|^2\right),\end{aligned}$$

where $\rho_i = \partial_i\rho$, $\left(\rho^{i\bar{j}}\right) = \left(\rho_{i\bar{j}}\right)^{-1}$, $\rho^i = \rho^{i\bar{j}}\rho_{\bar{j}}$ and $|d\rho|^2 = \rho^{i\bar{j}}\rho_i\rho_{\bar{j}}$. A straightforward consequence from these computations shows g_ρ is complete.

The Ricci curvature $\mathrm{Ric}(g_\rho)$ is given by

$$\begin{aligned}R_{i\bar{j}} &= -\partial_i\partial_{\bar{j}}\left(\log\left(\det\left(g_{k\bar{l}}\right)\right)\right) \\ &= -(n+1)g_{i\bar{j}} - \partial_i\partial_{\bar{j}}\log\left[\det\left(\rho_{k\bar{l}}\right)\left(-\rho + |d\rho|^2\right)\right].\end{aligned}$$

Note that $\det(\rho_{k\bar{l}})(-\rho+|d\rho|^2)$ is a positive C^{k-2} function defined on $\bar{\Omega}$. It can be computed that

$$\partial_i\partial_{\bar{j}}\log\left[\det(\rho_{k\bar{l}})\left(-\rho+|d\rho|^2\right)\right]$$

is a tensor whose norm tends to 0 near the boundary, so the Ricci curvature asymptotically tends to $-(n+1)$. Therefore, we have the following:

Corollary 4.4. Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^k-boundary for $k \geq 5$ and let g_ρ be the complete Kähler metric defined above. Suppose that g_ρ is a Kähler-Ricci soliton, then g_ρ is the unique Kähler-Einstein metric on strictly pseudoconvex domains constructed by Cheng-Yau.

5. The Bergman metric on uniformly squeezing domains

The uniform squeezing property was introduced by Liu et al. [LSY05] and by Yeung [Yeu09] independently in order to study canonical invariant metrics on complex manifolds.

Definition 5.1. A complex manifold M of dimension n is called uniformly squeezing if there exists $0 < r < R$ such that for any $p \in M$, there is a holomorphic map $f_p : M \to \mathbb{C}^n$ satisfying

(1) $f_p(p) = 0$;

(2) $f_p : M \to f_p(M)$ is biholomorphic;

(3) $B_r \subset f_p(M) \subset B_R$, where B_r and B_R are Euclidean balls in $\mathbb{C}^n$ with radius r and R centred at 0.

It is known that all bounded homogeneous domains, bounded domains that are covers of compact Kähler manifold, and strongly convex domains with C^2-boundary satisfy the uniformly squeezing property. It was proved in [Yeu09] that uniformly squeezing domains equipped with the Bergman metric or Kähler-Einstein metric have C^∞-bounded geometry and thus, the Ricci curvature is bounded.

In [DGZ12], Deng-Guan-Zhang introduced the concept of the squeezing function to study the geometric and analytic properties of uniformly squeezing domains.

Definition 5.2. Let Ω be a bounded domain in $\mathbb{C}^n$. For any $z \in \Omega$ and any holomorphic embedding $f : \Omega \to B_1$ with $f(z) = 0$, we set

$$s_\Omega(z, f) := \sup\{r; B_r \subset f(\Omega)\}.$$

The squeezing function of Ω is defined by

$$s_\Omega(z) := \sup_f\{s_\Omega(z, f)\}.$$

By definition, we see that $0 \leq s_\Omega(z) \leq 1$ and $s_\Omega(z)$ admits a positive lower bound if and only if Ω is uniformly squeezing. In [Zha15], Zhang established the curvature estimate of the Bergman metric in terms of the squeezing function.

Theorem 5.3 ([Zha15], Theorem 1.1)**.** Let Ω be a bounded domain in $\mathbb{C}^n$ and let s_Ω be the squeezing function. Denoting $H(z,W)$, $\mathrm{Ric}(z,W)$ and $R(z)$ by the holomorphic sectional curvature, the Ricci curvature and the scalar curvature at z in the direction W, respectively. Then we have

$$\begin{aligned} 2-2\frac{n+2}{n+1}s_\Omega^{-4n}(z) \le H(z,W) \le 2-2\frac{n+2}{n+1}s_\Omega^{4n}(z) \\ (n+1)-(n+2)s_\Omega^{-2n}(z) \le \mathrm{Ric}(z,W) \le (n+1)-(n+2)s_\Omega^{2n}(z), \\ n(n+1)-n(n+2)s_\Omega^{-2n}(z) \le R(z) \le n(n+1)-n(n+2)s_\Omega^{2n}(z). \end{aligned}$$

Thanks to the curvature estimate, we see that if the squeezing function is close to 1 outside a compact subset $K \subset\subset \Omega$, then the Ricci curvature is strictly negative on $\Omega \setminus K$. We can conclude the following:

Corollary 5.4. Let Ω be a bounded domain in $\mathbb{C}^n$ and let s_Ω be the squeezing function. Suppose there exists $0<\varepsilon<1$ such that

$$\left(\frac{n+1+\varepsilon}{n+2}\right)^{\frac{1}{2n}} \le s_\Omega \le 1,$$

in $\Omega \setminus K$ for some compact subset $K \subset\subset \Omega$. If g_B is a Kähler-Ricci soliton, then g_B is Kähler-Einstein.

It was proven in [DGZ16] that if Ω is a bounded strictly pseudoconvex domain with C^2-boundary, then $\lim_{z\to\partial\Omega} s_\Omega(z)=1$. Conversely, this statement is not always true, and we refer the interested reader to see [FW18] for a counterexample. In [Zim18], Zimmer proved that for a bounded convex domain Ω with C^∞-boundary, if $\lim_{z\to\partial\Omega} s_\Omega(z)=1$, then Ω is strictly pseudoconvex.

6. Example of model domains

6.1. Thullen domain in $\mathbb{C}^2$.

The Thullen domain

$$\Omega_m := \{(z,w)\in\mathbb{C}^2; |z|^2+|w|^{2m}<1\}$$

is a bounded pseudoconvex domain in $\mathbb{C}^2$ for any $m\ge 1$ with all boundary points are strictly pseudoconvex away from the boundary part $\{|z|=1\}$.

If $m=1$, the Thullen domain becomes the ball. For $m>1$, we observe that $-\log(1-|z|^2-|w|^{2m})$ is not strictly plurisubharmonic and therefore $g_{i\bar{j}}=-\partial_i\partial_{\bar{j}}\log(1-|z|^2-|w|^{2m})$ is not a Kähler metric. Instead, we consider another defining function $\rho(z,w)=(1-|z|^2)^{1/m}-|w|^2$, and the corresponding complete Kähler metric given by ρ introduced in [Seo12] is

$$g_m(z,w)=-\partial_i\partial_{\bar{j}}\log\rho(z,w).$$

The specific expression of g_m is

$$\frac{(1-|z|^2)^{\frac{1}{m}-2}}{m\rho^2}\begin{pmatrix}\rho+\frac{1}{m}|z|^2|w|^2 & w\bar{z}(1-|z|^2)\\ \bar{w}z(1-|z|^2) & m(1-|z|^2)^2\end{pmatrix},$$

and the determinant of g_m is

$$G := \det g_m = \frac{\frac{1}{m}\left(1-|z|^2\right)^{\frac{2}{m}-2}}{\rho^3}.$$

Then the Ricci curvature $\mathrm{Ric}(g_m) = -\partial_i\partial_{\bar{j}} \log G$ is computed by

$$\mathrm{Ric}(g_m) = -\frac{3(1-|z|^2)^{\frac{1}{m}-2}}{m\rho^2}\begin{pmatrix} \frac{2}{3}(m-1)\rho^2(1-|z|^2)^{-\frac{1}{m}} + \rho + \frac{1}{m}|z|^2|w|^2 & w\overline{z}(1-|z|^2) \\ \overline{w}z(1-|z|^2) & m(1-|z|^2)^2 \end{pmatrix}.$$

It is clear that g_m is Kähler-Einstein if and only if $m = 1$. By a straightforward computation, we can see the following.

Proposition 6.1. Suppose g_m is a Kähler-Ricci soliton, then $m = 1$.

Proof. Assume g_m is a Kähler-Ricci soliton associated with a real holomorphic vector field $X = X^i\partial_i + \overline{X}^i\partial_{\bar{i}}$ where X^i is a holomorphic function, that is

$$R_{i\bar{j}} + \nabla_i X_{\bar{j}} + \nabla_{\bar{j}} X_i = \lambda g_{i\bar{j}}, \tag{6.1}$$

for some constant λ and $i, j = 1$ or 2. We then compute

$$\nabla_i X_{\bar{j}} + \nabla_{\bar{j}} X_i = X^k\partial_k g_{i\bar{j}} + \overline{X}^l\partial_{\bar{l}} g_{i\bar{j}} + \partial_i X^k g_{k\bar{j}} + \partial_{\bar{j}}\overline{X}^l g_{i\bar{l}}.$$

First, fixing $i = j = 1$. Denote $\partial_1 = \partial_z$, $\partial_2 = \partial_w$, we have

$$\begin{aligned} X^k\partial_k g_{1\bar{1}} =& X^1\partial_1 g_{1\bar{1}} + X^2\partial_2 g_{1\bar{1}} \\ =& \frac{X^1\overline{z}}{m^3\rho^3\left(1-|z|^2\right)^{3-\frac{1}{m}}}\Big(2m^2\left(1-|z|^2\right)^{\frac{2}{m}} + \left((1-m)|z|^2 + 2m^2 - 2m\right)|w|^4 \\ & + \left(1-|z|^2\right)^{\frac{1}{m}}\left((m+1)|z|^2 - 4m^2 + 2m\right)|w|^2\Big) \\ & + \frac{X^2\overline{w}}{m^2\rho^3\left(1-|z|^2\right)^{2-\frac{1}{m}}}\left(\left(|z|^2 - m\right)|w|^2 + \left(1-|z|^2\right)^{\frac{1}{m}}\left(|z|^2 + m\right)\right), \end{aligned}$$

and

$$\begin{aligned} \partial_i X^k g_{k\bar{j}} =& \partial_1 X^1 g_{1\bar{1}} + \partial_1 X^2 g_{2\bar{1}} \\ =& \frac{\rho + \frac{1}{m}|z|^2|w|^2}{m\rho^2(1-|z|^2)^{2-\frac{1}{m}}}\cdot\partial_1 X^1 + \frac{z\overline{w}}{m\rho^2(1-|z|^2)^{1-\frac{1}{m}}}\cdot\partial_1 X^2. \end{aligned}$$

By combining the above computations and rearranging all terms in (6.1), we obtain

$$\begin{aligned}
&2(m-1)\rho^2(1-|z|^2)^{-\frac{1}{m}}+(3+\lambda)\left(\rho+\frac{1}{m}|z|^2|w|^2\right)\\
=&\frac{X^1\overline{z}+\overline{X}^1 z}{m^2\left(1-|z|^2\right)\rho}\Big(2m^2\left(1-|z|^2\right)^{\frac{2}{m}}+\left((1-m)\,|z|^2+2m^2-2m\right)|w|^4\\
&\qquad+\left(1-|z|^2\right)^{\frac{1}{m}}\left((m+1)\,|z|^2-4m^2+2m\right)|w|^2\Big)\\
&+\frac{X^2\overline{w}+\overline{X}^2 w}{m\rho}\left(\left(|z|^2-m\right)|w|^2+\left(1-|z|^2\right)^{\frac{1}{m}}\left(|z|^2+m\right)\right)\\
&+\left(\rho+\frac{1}{m}|z|^2|w|^2\right)\left(\partial_1 X^1+\partial_{\overline{1}}\overline{X}^1\right)+\left(1-|z|^2\right)\left(\partial_1 X^2\overline{w}z+\partial_{\overline{1}}\overline{X}^2 w\overline{z}\right).
\end{aligned}\tag{6.2}$$

Then, we take partial derivatives with respect to w and $\overline{w}$ twice (that is, applying $\partial_2^2\partial_{\overline{2}}^2$) to both sides of (6.2). By evaluating at $(z,0)$, we get

$$\frac{4|z|^2}{m(1-|z|^2)}\left(X^1\overline{z}+\overline{X}^1 z\right)+|z|^2\left(\partial_2 X^2+\partial_{\overline{2}}\overline{X}^2\right)=4m(m-1).\tag{6.3}$$

Next, fixing $i=j=2$. By taking the same approach as in the above computations, we have

$$4\left(X^1\overline{z}+\overline{X}^1 z\right)+m(1-|z|^2)\left(\partial_2 X^2+\partial_{\overline{2}}\overline{X}^2\right)=0,\tag{6.4}$$

which implies

$$\partial_2 X^2+\partial_{\overline{2}}\overline{X}^2=-\frac{4}{m(1-|z|^2)}\left(X^1\overline{z}+\overline{X}^1 z\right).\tag{6.5}$$

If $\partial_2 X^2+\partial_{\overline{2}}\overline{X}^2\neq 0$, then plugging (6.5) into (6.3), we obtain

$$4m(m-1)=0,$$

which yields $m=1$, since $m\geq 1$.

If $\partial_2 X^2+\partial_{\overline{2}}\overline{X}^2=0$, from (6.4), we have $X^1\overline{z}+\overline{X}^1 z=0$. Then, plugging into (6.3), we still have $m=1$. □

Remark 6.2. By the expression of $\mathrm{Ric}(g_m)$, we can see that $\mathrm{Ric}(g_m)$ is negatively pinched in Ω_m. Thus, Proposition 6.1 can be deduced from Theorem 1.1 directly.

We now consider the Bergman metric on the Thullen domain. The Bergman kernel of the Thullen domain Ω_m is given by

$$K(z,w)=\frac{m\pi^2}{m+1}\cdot\frac{(m+1)\left(1-|z|^2\right)^{\frac{1}{m}}-(m-1)|w|^2}{(m+1)\rho^3\left(1-|z|^2\right)^{2-\frac{1}{m}}}.$$

The Bergman metric g_B can then be computed. An interesting fact between the two metrics g_m and g_B on the Thullen domain is that they only coincide when Ω_m is the ball, that is $m=1$.

Proposition 6.3. Let $g_B=\lambda g_m$ for some $\lambda>0$, then $m=1$.

Proof. Assume $g_B = \lambda g_m$ for some $\lambda > 0$, then for some pluriharmonic functions h, we have

$$\log K + \lambda \log \rho = h. \tag{6.6}$$

Since the left-hand side of (6.6) only depends on $|z|$ and $|w|$, so does the right-hand side. Then by the maximum principle, we obtain that $h = C$ for some constant C. Evaluating at the origin, we get $C = \log\left(\frac{m\pi^2}{m+1}\right)$.

Thus, we have

$$\frac{(m+1)\left(1-|z|^2\right)^{\frac{1}{m}} - (m-1)|w|^2}{(m+1)\rho^{3-\lambda}\left(1-|z|^2\right)^{2-\frac{1}{m}}} = 1, \tag{6.7}$$

which implies that the left-hand side of (6.7) is independent of the value of z and w. When we approach the boundary, namely $|z|^2 + |w|^{2m} \to 1$, we note that the numerator of (6.7) does not always vanish, whereas the denominator always vanishes if $\lambda \neq 3$. That forces $\lambda = 3$. Let $(z, w) \to (0, 1) \in \partial\Omega_m$, we can verify that $m = 1$. □

As a significant example, the following computation verifies the recent work of Savale-Xiao [SX23] regarding Yau's question on bounded pseudoconvex domains with C^∞-boundary of finite type in dimension two.

Proposition 6.4. The Bergman metric g_B is Kähler-Einstein if and only if $m = 1$.

Proof. The "if" part follows from the case of the ball. For the "only if" part, we use the same notations as in [AS83]. It is enough to show this at $(0, w)$ with $|w| < 1$. Let

$$t = \frac{1-|w|^2}{1-r|w|^2}, \qquad |w| < 1,$$

where $r = (m-1)/(m+1)$. We note that $0 \leq r < 1$ and $0 < t \leq 1$. Then the Bergman metric is given by

$$g(0,w) = \begin{pmatrix} \alpha/(1+r)t & 0 \\ 0 & \beta(1-rt)^2/(1-r)^2t^2 \end{pmatrix}$$

and the Ricci curvature of the Bergman metric is given by

$$\mathrm{Ric}(0,w) = \begin{pmatrix} \frac{3\alpha^2\beta - 4A\beta - 2B\alpha}{\alpha\beta(1+r)t} & 0 \\ 0 & \frac{(1-rt)^2\left(3\alpha\beta^2 - 4C\alpha - 2B\beta\right)}{\alpha\beta(1-r)^2t^2} \end{pmatrix},$$

where

$$\begin{cases} \alpha = 3 + rt^2, \quad \beta = 3 - rt^2, \\ A = 6 + 4rt^2 + (1+r)rt^3, \\ B = 2\left(9 + 3rt^2 - 3(1+r)rt^3 + 2r^2t^4\right)/\left(3 + rt^2\right), \\ C = 3\left(6 - 6rt^2 + (1+r)rt^3\right)/\left(3 - rt^2\right). \end{cases} \tag{6.8}$$

Assume that the Bergman metric is Kähler-Einstein, that is

$$\mathrm{Ric}(0,w) = -g(0,w).$$

Consider the $(1,\bar{1})$-component of the Ricci curvature, after rearrangement we obtain

$$4\alpha^2\beta - 4A\beta - 2B\alpha = 0. \tag{6.9}$$

Plugging (6.8) into (6.9), we get

$$r^2[rt^2 - (1+r)t + 1] = 0.$$

Since $0 \leq r < 1$, we have $rt^2 - (1+r)t + 1 > 0$. This implies $r = 0$. □

It was proved in [Gon19] that the holomorphic bisectional curvature of g_B is negatively pinched in Ω_m. This implies that $\mathrm{Ric}(g_B)$ is also negatively pinched. Thus, we can characterize the Thullen domain if g_m or g_B is a Kähler-Ricci soliton.

Theorem 6.5. Let Ω_m be the bounded Thullen domain in $\mathbb{C}^2$. Then the following statements are equivalent.

(1) g_m (or g_B) is Kähler-Einstein;

(2) g_m (or g_B) is a Kähler-Ricci soliton;

(3) there is a constant $\lambda > 0$ such that $g_m = \lambda g_B$;

(4) $m = 1$ and Ω_m is the unit ball.

6.2. **Strictly pseudoconvex Hartogs domains.** Let $C \in (0,+\infty)$ and let $F : [0,B) \to (0,+\infty)$ be a continuous decreasing function that is smooth on $(0,B)$ satisfying

$$\frac{d}{dx}\left(\frac{xF'(x)}{F(x)}\right) < 0.$$

The bounded Hartogs domain $D_F \subset \mathbb{C}^n$ associated with the function F is a strictly pseudoconvex domain given by

$$D_F = \left\{(z_0, z_1, \ldots, z_{n-1}) \in \mathbb{C}^n; |z_0|^2 < x_0, |z_1|^2 + \cdots + |z_{n-1}|^2 < F\left(|z_0|^2\right)\right\}.$$

There is a corresponding complete Kähler metric defined by

$$g_F = -\partial_i\partial_{\bar{j}} \log\left(F\left(|z_0|^2\right) - |z_1|^2 - \cdots - |z_{n-1}|^2\right).$$

In [LZ10], the authors showed that the Ricci curvature of g_F is negatively pinched near the boundary. Consequently, we have the following:

Corollary 6.6 (Theorem 1.2, [LZ10])**.** Let $D_F \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex Hartogs domain equipped with a complete Kähler metric g_f defined above. Suppose g_F is a Kähler-Ricci soliton. Then g_F is Kähler-Einstein. Moreover, $F(x) = C_1 - C_2 x$ for some $C_1, C_2 > 0$, which implies that D_F is holomorphically isomorphic to an open subset of the complex hyperbolic space $\mathbb{H}^n$ via the map

$$\varphi : D_F \to \mathbb{H}^n, \quad (z_0, z_1, \ldots, z_{n-1}) \mapsto \left(\frac{z_0}{\sqrt{c_1/c_2}}, \frac{z_1}{\sqrt{c_1}}, \ldots, \frac{z_{n-1}}{\sqrt{c_1}}\right).$$

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