THE KÄHLER-RICCI SOLITON ON BOUNDED PSEUDOCONVEX DOMAINS

ZEHAO SHA

ABSTRACT. In this paper, we study Kähler–Ricci solitons on bounded pseudoconvex domains in $\mathbb{C}^n$. Under certain assumptions, we prove that such solitons must be Kähler–Einstein. We then focus on the Bergman metric and show that Bergman Kähler–Ricci solitons are Kähler-Einstein on bounded strictly pseudoconvex domains with C^∞-boundary. As a consequence, using Huang–Xiao's resolution of Cheng's conjecture, we obtain a soliton analogue of the ball characterization for bounded strictly pseudoconvex domains. Several examples are presented to illustrate the results.

1. INTRODUCTION

In Kähler geometry, a central theme is the search for canonical metrics in a given class. The most classical candidate is the Kähler–Einstein metric

$$\mathrm{Ric}(g) = \lambda g,$$

which has played a fundamental role since the work of Aubin [Aub78], Yau [Yau78] and many others. From the viewpoint of the Ricci flow, a natural enlargement of this notion is provided by Kähler–Ricci solitons. These are Kähler metrics satisfying

$$\mathrm{Ric}(g) + \mathcal{L}_X g = \lambda g, \tag{1.1}$$

where X is a real holomorphic vector field and $\lambda \in \mathbb{R}$. They arise as self-similar solutions to the Ricci flow, beginning with the work of Hamilton and Cao [Ham88, Cao96, Cao97]. A soliton is called trivial if X is Killing; in this case (1.1) reduces to the Kähler–Einstein equation.

On compact Kähler manifolds, the existence and rigidity of Kähler–Ricci solitons are strongly constrained by the sign of the first Chern class and by Futaki-type obstructions [Fut83]. In contrast, the non-compact case is considerably more flexible: complete non-trivial solitons do exist, and the problem of deciding when a soliton must be trivial becomes much more delicate.

The purpose of this paper is to study this rigidity problem on bounded pseudoconvex domains. Such domains form a basic class of non-compact complex manifolds with rich boundary geometry. In particular, every bounded pseudoconvex domain carries complete Kähler–Einstein metrics, as shown by Cheng–Yau [CY80] and Mok–Yau [MY83]. It is therefore natural to ask whether a Kähler–Ricci soliton structure on a bounded pseudoconvex domain can be genuinely non-trivial, or whether the geometry near the boundary forces the soliton to collapse to a Kähler–Einstein metric.

Key words and phrases. Kähler-Ricci soliton, Kähler-Einstein metric, Bergman metric, pseudoconvex domain.

Our first main result gives a general rigidity result for complete Kähler metrics on bounded pseudoconvex domains. The assumptions are designed to capture the asymptotically negative Ricci curvature behavior that appears naturally near the boundary for many complete metrics.

Theorem A. Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with C^2-boundary, and let g be a complete Kähler metric on Ω of C^1 quasi-bounded geometry. Assume that there exists a compact subset $K \Subset \Omega$ such that

$$-C_2 g \leq \mathrm{Ric}(g) \leq -C_1 g \qquad \text{on } \Omega \setminus K,$$

for constants $C_2 \geq C_1 > 0$. If g is a Kähler–Ricci soliton with real holomorphic vector field X, and if the dual one-form of X with respect to g is closed, then $X = 0$. In particular, g is Kähler–Einstein.

The proof is based on a Bochner formula for real holomorphic vector fields, together with the Omori–Yau maximum principle. The asymptotic negativity of the Ricci tensor first gives a global bound for the soliton vector field. The soliton equation and the Bochner identity then force the maximum of $|X|^2$ to vanish. The closedness assumption on the dual one-form is automatic in the gradient case, but is slightly weaker than requiring a globally defined soliton potential.

We next turn to the Bergman metric. For a bounded domain $\Omega \subset \mathbb{C}^n$, the Bergman kernel $K_\Omega(z, \bar{z})$ defines the Bergman metric

$$g_B = \sqrt{-1}\, \partial\bar{\partial} \log K_\Omega(z, \bar{z}),$$

which is biholomorphically invariant. This metric is one of the most important intrinsic Kähler metrics in several complex variables, and its curvature reflects fine boundary geometry of the domain.

The relation between the Bergman metric and the complete Kähler–Einstein metric has been a central theme since the work of Cheng–Yau [CY80] and Mok–Yau [MY83] on complete Kähler–Einstein metrics on bounded pseudoconvex domains. Yau asked, in broad form, whether a bounded pseudoconvex domain with Kähler–Einstein Bergman metric should be homogeneous [SY94]. On strictly pseudoconvex domains with C^∞-boundary, Cheng formulated the sharper conjecture that such domain has Bergman Kähler–Einstein metric if and only if it is biholomorphic to the unit ball [Che]. This conjecture was proved in all dimensions by Huang and Xiao [HX21].

The soliton analogue of this problem is the following:

Question 1.1. Can the Bergman metric of a bounded pseudoconvex domain carry a non-trivial Kähler–Ricci soliton structure? In particular, on a smoothly bounded strictly pseudoconvex domain, does the Bergman Kähler–Ricci soliton condition characterize the unit ball?

Our second main result answers this question on strictly pseudoconvex domains.

Theorem B. Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^∞-boundary, and let g_B be its Bergman metric. If g_B is a Kähler–Ricci soliton, then g_B is Kähler–Einstein. Moreover, Ω is biholomorphic to the unit ball.

The proof of Theorem B uses a different and more intrinsic mechanism than that of Theorem A. The boundary curvature estimates for the Bergman metric imply that $\mathrm{Ric}(g_B) + g_B$ is bounded with respect to g_B. Hence the soliton equation gives a bound for $\mathcal{L}_X g_B$. In particular, X is complete. Since X is real holomorphic, its flow is a one-parameter group of biholomorphisms of Ω. By the biholomorphic invariance of the Bergman metric, this flow acts by isometries of g_B, hence $\mathcal{L}_X g_B = 0$. Thus the soliton is trivial and g_B is Kähler–Einstein. The final ball characterization then follows from Huang–Xiao's theorem [HX21, Theorem 1.1].

Theorem A leaves open a natural broader problem: whether bounded pseudoconvex domains with C^2-boundary can support genuinely non-trivial Kähler–Ricci solitons. Our result gives a rigidity answer under the additional assumption that the Ricci curvature of the complete Kähler metric is asymptotically negative near the boundary. In this setting, the soliton vector field is forced to vanish, provided its dual one-form is closed. This closedness condition is automatic for gradient Kähler–Ricci solitons, although the converse may fail because of global topological obstructions.

The asymptotic Ricci-negativity assumption is nevertheless a genuine restriction. Not every complete Kähler metric on a bounded pseudoconvex domain is expected to satisfy it; for instance, the boundary behavior of the Bergman metric on general pseudoconvex domains can be considerably more subtle, see [KY96]. Thus it remains an interesting problem to decide whether every Kähler–Ricci soliton on a bounded pseudoconvex domain with C^2-boundary must be trivial, or whether one can construct a non-trivial example outside the scope of Theorem A.

Acknowledgements. The author is deeply grateful to his advisors Gérard Besson and Hervé Gaussier for their encouragement, guidance, and many helpful discussions. He also thanks Alix Deruelle and Vincent Koziarz for valuable comments on an earlier version of this manuscript.

2. Preliminaries

2.1. **Conventions.** Let (M^n, g) be a complete Kähler manifold. In local holomorphic coordinates $(z^1, \dots, z^n)$, we write

$$g = g_{i\bar{j}}\, dz^i \otimes d\bar{z}^j, \qquad (g^{i\bar{j}}) = (g_{i\bar{j}})^{-1},$$

and we use the Einstein summation convention. All norms, inner products, and contractions are taken with respect to g.

The Ricci tensor is given by

$$\mathrm{Ric}(g) = R_{i\bar{j}}\, dz^i \otimes d\bar{z}^j, \qquad R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}}),$$

and the scalar curvature is

$$R = g^{i\bar{j}} R_{i\bar{j}}.$$

We denote by ∇ the Levi–Civita connection of g. For a smooth real-valued function f, we use the convention

$$\Delta f = g^{i\bar{j}} \partial_i \partial_{\bar{j}} f.$$

Thus Δ is the complex Laplacian. Since g is Kähler, the mixed Christoffel symbols vanish in holomorphic coordinates, and hence

$$\nabla_i \nabla_{\bar{j}} f = \partial_i \partial_{\bar{j}} f.$$

2.2. **Kähler–Ricci solitons.** A Kähler metric g is Kähler–Einstein if

$$\mathrm{Ric}(g) = \lambda g \tag{2.1}$$

for some constant $\lambda \in \mathbb{R}$.

Let X be a real vector field on M. We say that X is real holomorphic if its $(1,0)$-part

$$X^{1,0} := \frac{1}{2}(X - \sqrt{-1} JX)$$

is a holomorphic vector field. A Kähler metric g is called a Kähler–Ricci soliton if there exist a real holomorphic vector field X and a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(g) + \mathcal{L}_X g = \lambda g. \tag{2.2}$$

The soliton is called shrinking, steady, or expanding according as $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$. It is called trivial if X is Killing. In that case $\mathcal{L}_X g = 0$, and (2.2) reduces to the Kähler–Einstein equation (2.1).

If there exists a smooth real-valued function f such that $X = \frac{1}{2}\nabla f$, then g is called a gradient Kähler–Ricci soliton, and f is called a potential function. In this case the dual one-form $X^\flat := g(X, \cdot)$ satisfies $X^\flat = \frac{1}{2}\, df$, and hence is closed.

2.3. **Quasi-bounded geometry and the Omori–Yau maximum principle.** To establish our main result, we first introduce the concept of (quasi-)bounded geometry.

Definition 2.1. Let (M, g) be a complete Kähler manifold and let $k \geq 0$. We say that (M, g) has C^k-quasi-bounded geometry if, for each $0 \leq \ell \leq k$, there exists a constant $C_\ell > 0$ such that

$$\sup_M |\nabla^\ell \mathrm{Rm}| \leq C_\ell.$$

If, in addition, the injectivity radius of g is bounded from below by a positive constant, then g is said to have C^k-bounded geometry.

We shall use the following form of the Omori–Yau maximum principle [Omo67, Yau75].

Proposition 2.2 (Omori–Yau maximum principle)**.** Let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded from below. If $u \in C^2(M)$ satisfies $\sup_M u < +\infty$, then there exists a sequence $\{p_j\} \subset M$ such that

$$u(p_j) \to \sup_M u, \qquad |\nabla u|(p_j) \to 0, \qquad \limsup_{j \to \infty} \Delta u(p_j) \leq 0.$$

3. Proof of Theorem A

Let $\Omega \subset \mathbb{C}^n$ and g be as in Theorem A. Assume that g is a Kähler-Ricci soliton for a real holomorphic vector field X. We first note that the soliton must be expanding. Indeed, by the Ricci pinching assumption,

$$R \leq -nC_1 \qquad \text{on } \Omega \setminus K.$$

Thus the scalar curvature is negative somewhere. On the other hand, the sharp lower bounded estimates of the scalar curvature for complete shrinking and steady Ricci solitons gives $R \geq 0$ (see, e.g. [Cho23]). Hence $\lambda < 0$, and we may and will assume throughout the proof that

$$\mathrm{Ric}(g) + \mathcal{L}_X g = -g. \tag{3.1}$$

We shall use the closedness assumption on the dual one-form $X^\flat = g(X, \cdot)$ in the following form. Since $dX^\flat = 0$, for any vector fields Y, Z,

$$(\nabla_Y X^\flat)(Z) = (\nabla_Z X^\flat)(Y).$$

In local holomorphic coordinates this gives, in particular,

$$\nabla_i X_{\bar{j}} = \nabla_{\bar{j}} X_i. \tag{3.2}$$

We also record the following uniqueness observation.

Proposition 3.1. Under the assumptions of Theorem A, the real holomorphic soliton vector field whose dual one-form is closed is unique for a fixed soliton constant.

Proof. Suppose that X and $\widetilde{X}$ are two such vector fields with the same soliton constant, and set $Y = X - \widetilde{X}$. Subtracting the two soliton equations gives

$$\mathcal{L}_Y g = 0.$$

Moreover, Y is real holomorphic and $Y^\flat$ is closed. Hence, in local coordinates,

$$\nabla_j Y_i = 0, \qquad \nabla_{\bar{j}} Y_i = 0.$$

Applying $g^{k\bar{j}} \nabla_k$ to the second identity and commuting covariant derivatives, we obtain

$$0 = g^{k\bar{j}} \nabla_k \nabla_{\bar{j}} Y_i = -R_{i\bar{l}} Y^{\bar{l}}.$$

Thus $\mathrm{Ric}(Y, Y) = 0$. Since $\mathrm{Ric}(g) \leq -C_1 g$ on $\Omega \setminus K$, it follows that $Y = 0$ on $\Omega \setminus K$. By the identity theorem for holomorphic vector fields, $Y \equiv 0$ on Ω. □

Next we recall the Bochner formula needed below.

Lemma 3.2. Let g be a Kähler metric and let X be a real holomorphic vector field. Then

$$\Delta |X|^2 = |\nabla X|^2 - \mathrm{Ric}(X, X). \tag{3.3}$$

Proof. We compute this formula in local coordinates. Since X is real holomorphic, $\nabla_i X_j = 0$. Therefore

$$\Delta|X|^2 = \nabla^i\nabla_i(X^kX_k) = \nabla^i X_k \nabla_i X^k + X_k \nabla^i\nabla_i X^k.$$

Commuting derivatives gives

$$\nabla^i\nabla_i X^k = -R_\ell{}^k X^\ell,$$

and hence

$$\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ric}(X,X),$$

which completes the proof. □

We now prove Theorem A.

Proof of Theorem A. By the normalized soliton equation (3.1), we have in local coordinates

$$R_{i\bar{j}} + \nabla_i X_{\bar{j}} + \nabla_{\bar{j}} X_i = -g_{i\bar{j}}. \tag{3.4}$$

Using (3.2), this becomes

$$R_{i\bar{j}} + 2\nabla_{\bar{j}} X_i = -g_{i\bar{j}}.$$

Applying $g^{k\bar{j}}\nabla_k$ and using the contracted Bianchi identity gives

$$\begin{aligned} 0 &= g^{k\bar{j}}\nabla_k R_{i\bar{j}} + 2g^{k\bar{j}}\nabla_k\nabla_{\bar{j}} X_i \\ &= \nabla_i R + 2g^{k\bar{j}}\left(\nabla_{\bar{j}}\nabla_k X_i - R_{k\bar{j}i\bar{l}}X^{\bar{l}}\right). \end{aligned}$$

Since X is real holomorphic, $\nabla_k X_i = 0$. Hence

$$\nabla_i R = 2R_{i\bar{l}}X^{\bar{l}}. \tag{3.5}$$

Equivalently,

$$g(\nabla R, \cdot) = 2\,\mathrm{Ric}(X, \cdot).$$

We first prove that X is bounded. On $\Omega \setminus K$, the Ricci pinching assumption gives

$$\mathrm{Ric}(X,X) \le -C_1|X|^2.$$

Taking the pairing of (3.5) with X, we obtain

$$g(\nabla R, X) = 2\,\mathrm{Ric}(X,X) \le -2C_1|X|^2.$$

Therefore, on $\Omega \setminus K$,

$$2C_1|X|^2 \le -g(\nabla R, X) \le |\nabla R|\,|X|.$$

It follows that

$$|X| \le \frac{|\nabla R|}{2C_1} \qquad \text{on } \Omega \setminus K.$$

Since C^1-quasi-bounded geometry implies $\sup_\Omega |\nabla R| < +\infty$, and since X is bounded on the compact set K, we conclude that

$$\sup_\Omega |X| < +\infty. \tag{3.6}$$

Next, evaluate the normalized soliton equation (3.1) on (X,X):

$$\mathrm{Ric}(X,X) + (\mathcal{L}_X g)(X,X) = -|X|^2.$$

Since

$$(\mathcal{L}_X g)(X,X) = \nabla_X |X|^2,$$

we get

$$\mathrm{Ric}(X,X) = -\nabla_X |X|^2 - |X|^2. \tag{3.7}$$

Combining (3.3) and (3.7), we find

$$\Delta |X|^2 = |\nabla X|^2 + \nabla_X |X|^2 + |X|^2 \geq \nabla_X |X|^2 + |X|^2.$$

Now apply the Omori–Yau maximum principle to the bounded function

$$u := |X|^2.$$

There exists a sequence $\{p_j\} \subset \Omega$ such that

$$u(p_j) \to \sup_\Omega u, \qquad |\nabla u|(p_j) \to 0, \qquad \limsup_{j\to\infty} \Delta u(p_j) \leq 0.$$

Using the boundedness of X, we have

$$\nabla_X u(p_j) \to 0.$$

Therefore,

$$0 \geq \limsup_{j\to\infty} \Delta u(p_j) \geq \lim_{j\to\infty} \nabla_X u(p_j) + \lim_{j\to\infty} u(p_j) = \sup_\Omega u.$$

Hence $\sup_\Omega |X|^2 = 0$, so $X \equiv 0$. Consequently, g is Kähler–Einstein. □

4. Two Kähler metrics on strictly pseudoconvex domains

In this section we discuss two natural types of complete Kähler metrics on bounded pseudoconvex domains.

4.1. The Bergman metric. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and let

$$A^2(\Omega) = \{f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 \, dV < +\infty\}$$

be its Bergman space. Denote by $K_\Omega(z,w)$ the Bergman kernel and write

$$K_\Omega(z) := K_\Omega(z,z).$$

The Bergman metric is the Kähler metric

$$\omega_B = \sqrt{-1}\,\partial\bar\partial \log K_\Omega(z), \qquad g_B(\cdot,\cdot) = \omega_B(\cdot, J\cdot).$$

Note that the Bergman metric is biholomorphically invariant. In particular, every automorphism of Ω acts by isometries of g_B.

We first record a simple completeness criterion for vector fields.

Lemma 4.1. Let (M,g) be a complete Riemannian manifold and let X be a smooth vector field on M. If

$$\|\mathcal{L}_X g\|_g \leq C$$

for some constant $C > 0$, then X is complete.

Proof. Let $\gamma : (a, b) \to M$ be a maximal integral curve of X, so that $\dot\gamma(t) = X(\gamma(t))$. Along γ, we have

$$\frac{d}{dt}|X|_g^2(\gamma(t)) = X\big(g(X, X)\big)(\gamma(t)) = (\mathcal{L}_X g)(X, X)(\gamma(t)).$$

By the assumed bound,

$$\left|\frac{d}{dt}|X|_g^2(\gamma(t))\right| \le C|X|_g^2(\gamma(t)).$$

Gronwall's inequality therefore implies that $|X|_g(\gamma(t))$ remains bounded up to any finite endpoint of the maximal interval of existence. Suppose, for contradiction, that $b < +\infty$. Fix $t_0 \in (a, b)$. Since $|X|_g(\gamma(t))$ is bounded on $[t_0, b)$, the curve γ has finite length on $[t_0, b)$. In particular, $\gamma(t)$ is a Cauchy curve as $t \to b^-$. Since (M, g) is complete, there exists a point $p \in M$ such that

$$\gamma(t) \to p \qquad \text{as } t \to b^-.$$

By the smoothness of X, the ordinary differential equation $\dot\eta = X(\eta)$ has a local solution starting from p. Thus γ can be extended beyond b, contradicting the maximality of (a, b). Therefore $b = +\infty$. Applying the same argument backward in time gives $a = -\infty$. Hence X is complete. □

We now prove the Bergman rigidity theorem.

Theorem 4.2. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain whose Bergman metric g_B is complete. Suppose that

$$\mathrm{Ric}(g_B) + \mathcal{L}_X g_B = \lambda g_B$$

for a real holomorphic vector field X and a constant $\lambda \in \mathbb{R}$. Assume moreover that

$$\|\mathrm{Ric}(g_B) - \lambda g_B\|_{g_B} \le C$$

for some constant $C > 0$. Then X is Killing. Consequently, g_B is Kähler–Einstein.

Proof. It follows from the soliton equation that

$$\|\mathcal{L}_X g_B\|_{g_B} = \|\lambda g_B - \mathrm{Ric}(g_B)\|_{g_B} \le C.$$

Since g_B is complete, Lemma 4.1 implies that X is complete.

Let Φ_t be the complete flow of X. Since X is real holomorphic, Φ_t is a one-parameter group of biholomorphisms of Ω. By the biholomorphic invariance of the Bergman metric,

$$\Phi_t^* g_B = g_B \qquad \text{for all } t \in \mathbb{R}.$$

Differentiating at $t = 0$, we obtain

$$\mathcal{L}_X g_B = 0.$$

The soliton equation therefore reduces to

$$\mathrm{Ric}(g_B) = \lambda g_B.$$

Thus g_B is Kähler–Einstein. □

On bounded strictly pseudoconvex domains with C^∞-boundary, the Bergman metric is complete and has C^∞-bounded geometry near the boundary; see [GKK11]. In particular, we shall use below only the following consequence, see also [KY96]:

$$\|\operatorname{Ric}(g_B)+g_B\|_{g_B}\le C \qquad \text{on } \Omega. \tag{4.1}$$

Corollary 4.3. Let $\Omega\subset\mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^∞-boundary, and let g_B be its Bergman metric. Assume that g_B is a Kähler–Ricci soliton for a real holomorphic vector field X. Then g_B is Kähler–Einstein. Moreover, Ω is biholomorphic to the unit ball.

Proof. Let λ be the soliton constant. Since

$$\|\operatorname{Ric}(g_B)+g_B\|_{g_B}\le C,$$

we also have

$$\|\operatorname{Ric}(g_B)-\lambda g_B\|_{g_B}\le C+|\lambda+1|\,\|g_B\|_{g_B}=C+|\lambda+1|\sqrt{n}.$$

Hence the hypotheses of Theorem 4.2 are satisfied with λ. Therefore g_B is Kähler–Einstein. The biholomorphic ball characterization then follows from Huang–Xiao's theorem [HX21, Theorem 1.1]. □

4.2. **Metrics induced by defining functions.** We now turn to a different class of complete Kähler metrics. Let $\Omega\subset\mathbb{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary, and let $\rho\in C^\infty(\overline{\Omega})$ be a strictly plurisubharmonic defining function:

$$\Omega=\{\rho<0\},\qquad d\rho\neq 0 \text{ on } \partial\Omega,\qquad \sqrt{-1}\,\partial\bar\partial\rho>0 \text{ near } \overline{\Omega}.$$

Set

$$\phi=-\log(-\rho),\qquad \omega_\rho=\sqrt{-1}\,\partial\bar\partial\phi,\qquad g_\rho(\cdot,\cdot)=\omega_\rho(\cdot,J\cdot).$$

Then g_ρ is a complete Kähler metric on Ω.

A direct computation gives

$$g_{i\bar j}=\frac{\rho_{i\bar j}}{-\rho}+\frac{\rho_i\rho_{\bar j}}{\rho^2},\qquad g^{i\bar j}=(-\rho)\left(\rho^{i\bar j}+\frac{\rho^i\rho^{\bar j}}{\rho-|d\rho|^2}\right)$$

and

$$\begin{aligned}\det\left(g_{i\bar j}\right)&=\left(-\frac{1}{\rho}\right)^n\det\left(\rho_{i\bar j}-\frac{\rho_i\rho_{\bar j}}{\rho}\right)\\&=\left(\frac{1}{\rho}\right)^{n+1}\det\left(\rho_{i\bar j}\right)\left(-\rho+|d\rho|^2\right),\end{aligned}$$

where $\rho_i=\partial_i\rho$, $\left(\rho^{i\bar j}\right)=\left(\rho_{i\bar j}\right)^{-1}$, $\rho^i=\rho^{i\bar j}\rho_{\bar j}$ and $|d\rho|^2=\rho^{i\bar j}\rho_i\rho_{\bar j}$. A straightforward consequence from these computations shows g_ρ is complete and has C^∞ quasi-bounded geometry [CY80].

The Ricci curvature $\mathrm{Ric}(g_\rho)$ is given by

$$\begin{aligned} R_{i\bar{j}} &= -\partial_i \partial_{\bar{j}} \left(\log\left(\det\left(g_{k\bar{l}}\right)\right)\right) \\ &= -(n+1) g_{i\bar{j}} - \partial_i \partial_{\bar{j}} \log\left[\det\left(\rho_{k\bar{l}}\right)\left(-\rho + |d\rho|^2\right)\right]. \end{aligned}$$

Note that $\det\left(\rho_{k\bar{l}}\right)\left(-\rho + |d\rho|^2\right)$ is a positive C^∞ function defined on $\bar{\Omega}$. A standard boundary computation shows that

$$\partial_i \partial_{\bar{j}} \log\left[\det\left(\rho_{k\bar{l}}\right)\left(-\rho + |d\rho|^2\right)\right]$$

has g_ρ-norm tending to 0 as one approaches $\partial\Omega$, so the Ricci curvature asymptotically tends to $-(n+1)$. As a consequence of Theorem A, we obtain the following rigidity statement.

Corollary 4.4. Let $\Omega \subset \mathbb{C}^n$ and g_ρ be as above. If g_ρ is a gradient Kähler–Ricci soliton, then g_ρ is Kähler–Einstein.

5. Examples

5.1. **Uniformly squeezing domains.** The uniform squeezing property was introduced independently by Liu–Sun–Yau [LSY05] and Yeung [Yeu09] in the study of canonical invariant metrics on complex manifolds. It provides a flexible framework in which the Bergman metric, the Kobayashi metric, and the complete Kähler–Einstein metric enjoy uniform geometric control.

Definition 5.1. A complex manifold M of dimension n is called uniformly squeezing if there exist constants $0 < r < R$ such that for every $p \in M$, there is a holomorphic map $f_p : M \longrightarrow \mathbb{C}^n$ satisfying:

(1) $f_p(p) = 0$;

(2) $f_p : M \to f_p(M)$ is biholomorphic;

(3) $B_r \subset f_p(M) \subset B_R$, where B_r and B_R are Euclidean balls in $\mathbb{C}^n$ centered at 0.

Typical examples include bounded homogeneous domains, bounded domains which are covering spaces of compact Kähler manifolds, and strongly convex domains with C^2-boundary. A fundamental result of Yeung [Yeu09] shows that on uniformly squeezing domains, the Bergman metric and the complete Kähler–Einstein metric have C^∞-bounded geometry. In particular, the Bergman metric is complete and its curvature tensor, together with all covariant derivatives, is uniformly bounded. Combining this with Theorem 4.2 gives the following immediate consequence.

Corollary 5.2. Let $\Omega \subset \mathbb{C}^n$ be a bounded uniformly squeezing domain, and let g_B be its Bergman metric. Suppose that

$$\mathrm{Ric}(g_B) + \mathcal{L}_X g_B = \lambda g_B$$

for a real holomorphic vector field X and a constant $\lambda \in \mathbb{R}$. Then X is Killing. In particular, g_B is Kähler–Einstein.

5.2. **Thullen domains in $\mathbb{C}^2$.** Let $m \in \mathbb{N}$, $m \geq 1$, and consider the Thullen domain

$$\Omega_m := \{(z,w) \in \mathbb{C}^2 : \ |z|^2 + |w|^{2m} < 1\}.$$

It is a bounded pseudoconvex complete Reinhardt domain. When $m = 1$, Ω_m is the unit ball. When $m > 1$, the boundary is weakly pseudoconvex precisely along

$$\{(z,0) : |z| = 1\} \subset \partial\Omega_m.$$

For $m > 1$, the function

$$-\log(1 - |z|^2 - |w|^{2m})$$

does not define a Kähler metric on all of Ω_m. We instead use the defining function

$$\rho(z,w) := (1 - |z|^2)^{1/m} - |w|^2$$

and the complete Kähler metric $g_m = -\partial\bar\partial \log \rho$ introduced in this context in [Seo12]. Explicitly,

$$g_m = \frac{(1-|z|^2)^{\frac{1}{m}-2}}{m\rho^2} \begin{pmatrix} \rho + \frac{1}{m}|z|^2|w|^2 & w\bar z(1-|z|^2) \\ \bar w z(1-|z|^2) & m(1-|z|^2)^2 \end{pmatrix}.$$

Its determinant is

$$\det(g_m) = \frac{1}{m} \frac{(1-|z|^2)^{\frac{2}{m}-2}}{\rho^3}.$$

Consequently,

$$\mathrm{Ric}(g_m) = -\partial\bar\partial \log\det(g_m),$$

and a direct computation gives

$$\mathrm{Ric}(g_m) = -\frac{3(1-|z|^2)^{\frac{1}{m}-2}}{m\rho^2} \begin{pmatrix} \frac{2}{3}(m-1)\rho^2(1-|z|^2)^{-\frac{1}{m}} + \rho + \frac{1}{m}|z|^2|w|^2 & w\bar z(1-|z|^2) \\ \bar w z(1-|z|^2) & m(1-|z|^2)^2 \end{pmatrix}.$$

In particular, g_m is Kähler–Einstein if and only if $m = 1$.

We first show that g_m admits no non-trivial Kähler–Ricci soliton structure.

Proposition 5.3. If g_m is a Kähler–Ricci soliton on Ω_m, then $m = 1$.

Proof. Assume that

$$\mathrm{Ric}(g_m) + \mathcal{L}_X g_m = \lambda g_m$$

for a real holomorphic vector field

$$X = X^1\partial_z + X^2\partial_w + \overline{X^1}\partial_{\bar z} + \overline{X^2}\partial_{\bar w},$$

where X^1 and X^2 are holomorphic functions on Ω_m.

We use the local expression

$$(\mathcal{L}_X g_m)_{i\bar j} = X^k\partial_k(g_m)_{i\bar j} + \overline{X^l}\partial_{\bar l}(g_m)_{i\bar j} + (\partial_i X^k)(g_m)_{k\bar j} + (\partial_{\bar j}\overline{X^l})(g_m)_{i\bar l}.$$

Applying this to the $(1,\bar 1)$-component of the soliton equation, then applying $\partial_w^2\partial_{\bar w}^2$ and evaluating at $(z,0)$, one obtains

$$\frac{4|z|^2}{m(1-|z|^2)}\left(X^1\bar z + \overline{X^1}z\right) + |z|^2\left(\partial_w X^2 + \partial_{\bar w}\overline{X^2}\right) = 4m(m-1). \tag{5.1}$$

Similarly, applying the same procedure to the $(2,\bar{2})$-component gives

$$4\big(X^1\bar{z}+\overline{X^1}z\big)+m(1-|z|^2)\big(\partial_w X^2+\partial_{\bar{w}}\overline{X^2}\big)=0. \tag{5.2}$$

All quantities in (5.1) and (5.2) are evaluated at $(z,0)$.

From (5.2),

$$\partial_w X^2+\partial_{\bar{w}}\overline{X^2}=-\frac{4}{m(1-|z|^2)}\big(X^1\bar{z}+\overline{X^1}z\big).$$

Substituting this into (5.1) gives

$$4m(m-1)=0.$$

Since $m\geq 1$, we conclude that $m=1$. □

We next compare g_m with the Bergman metric. The Bergman kernel of Ω_m is explicitly given by

$$K_{\Omega_m}(z,w)=\frac{m}{\pi^2}\,\frac{(m+1)(1-|z|^2)^{1/m}-(m-1)|w|^2}{\rho^3(1-|z|^2)^{2-\frac{1}{m}}}. \tag{5.3}$$

This normalization agrees with the standard Bergman kernel of the unit ball when $m=1$.

Proposition 5.4. If there exists a constant $c>0$ such that $g_B=c\,g_m$, then $m=1$.

Proof. Assume $g_B=cg_m$. Since

$$g_B=\partial\bar{\partial}\log K_{\Omega_m},\qquad g_m=-\partial\bar{\partial}\log\rho,$$

there exists a pluriharmonic function h such that

$$\log K_{\Omega_m}+c\log\rho=h.$$

The left-hand side depends only on $|z|$ and $|w|$. Hence h is a smooth rotation-invariant pluriharmonic function on the complete Reinhardt domain Ω_m. Therefore h is constant.

Using (5.3), we get

$$\frac{(m+1)(1-|z|^2)^{1/m}-(m-1)|w|^2}{\rho^{3-c}(1-|z|^2)^{2-\frac{1}{m}}}=C$$

for some constant $C>0$. Approaching a boundary point at which the numerator does not vanish forces $c=3$. Hence

$$\frac{(m+1)(1-|z|^2)^{1/m}-(m-1)|w|^2}{(1-|z|^2)^{2-\frac{1}{m}}}$$

is constant. Setting $w=0$, this becomes

$$(m+1)(1-|z|^2)^{\frac{2}{m}-2}=\text{constant}.$$

Therefore $\frac{2}{m}-2=0$, and hence $m=1$. □

We also have the following consequence of the explicit Bergman kernel computation.

Proposition 5.5. The Bergman metric g_B of Ω_m is Kähler–Einstein if and only if $m=1$.

Proof. The "if" part follows from the case of the ball. For the "only if" part, we use the same notations as in [AS83]. It is enough to show this at $(0, w)$ with $|w| < 1$. Let

$$t = \frac{1 - |w|^2}{1 - r|w|^2}, \qquad |w| < 1,$$

where $r = (m-1)/(m+1)$. Note that $0 \le r < 1$ and $0 < t \le 1$. Then the Bergman metric is given by

$$g(0, w) = \begin{pmatrix} \alpha/(1+r)t & 0 \\ 0 & \beta(1-rt)^2/(1-r)^2t^2 \end{pmatrix}$$

and the Ricci curvature of the Bergman metric is given by

$$\mathrm{Ric}(0, w) = \begin{pmatrix} \frac{3\alpha^2\beta - 4A\beta - 2B\alpha}{\alpha\beta(1+r)t} & 0 \\ 0 & \frac{(1-rt)^2\left(3\alpha\beta^2 - 4C\alpha - 2B\beta\right)}{\alpha\beta(1-r)^2t^2} \end{pmatrix},$$

where

$$\begin{cases} \alpha = 3 + rt^2, \quad \beta = 3 - rt^2, \\ A = 6 + 4rt^2 + (1+r)rt^3, \\ B = 2\left(9 + 3rt^2 - 3(1+r)rt^3 + 2r^2t^4\right) / \left(3 + rt^2\right), \\ C = 3\left(6 - 6rt^2 + (1+r)rt^3\right) / \left(3 - rt^2\right). \end{cases} \tag{5.4}$$

Assume that the Bergman metric is Kähler-Einstein, that is

$$\mathrm{Ric}(0, w) = -g(0, w).$$

Consider the $(1, \bar{1})$-component of the Ricci curvature, after rearrangement we obtain

$$4\alpha^2\beta - 4A\beta - 2B\alpha = 0. \tag{5.5}$$

Plugging (5.4) into (5.5), we get

$$r^2[rt^2 - (1+r)t + 1] = 0.$$

Since $0 \le r < 1$, for $0 < t < 1$, we have

$$rt^2 - (1+r)t + 1 = (1-t)(1-rt) > 0.$$

Since the equation holds for all $|w| < 1$, we may choose $w \neq 0$, hence $0 < t < 1$. Therefore $r = 0$. □

It was proved in [Gon19] that the holomorphic bisectional curvature of the Bergman metric on Ω_m is negatively pinched. In particular, $\|\mathrm{Ric}(g_B) - \lambda g_B\|_{g_B}$ is bounded for every constant λ. Therefore, Theorem 4.2 applies to g_B. Combining the preceding results, we obtain the following characterization.

Theorem 5.6. Let

$$\Omega_m = \{(z, w) \in \mathbb{C}^2 : \ |z|^2 + |w|^{2m} < 1\}, \qquad m \in \mathbb{N}_{\ge 1},$$

be the Thullen domain. Then the following are equivalent:

(1) $m = 1$, equivalently Ω_m is the unit ball;

(2) g_m is Kähler–Einstein, equivalently g_m is a Kähler–Ricci soliton;

(3) g_B is Kähler–Einstein, equivalently g_B is a Kähler–Ricci soliton;

(4) there exists a constant $c > 0$ such that $g_B = cg_m$.

Proof. The equivalence between (1) and the Kähler–Einstein condition for g_m follows from the explicit formula for $\mathrm{Ric}(g_m)$, while Proposition 5.3 shows that a Kähler–Ricci soliton structure on g_m forces $m = 1$. Thus (1) and (2) are equivalent.

Similarly, Proposition 5.5 gives the equivalence between (1) and the Kähler–Einstein condition for g_B. If g_B is a Kähler–Ricci soliton, then Theorem 4.2, together with the curvature pinching result of [Gon19], implies that g_B is Kähler–Einstein. Hence (1) and (3) are equivalent.

Finally, Proposition 5.4 shows that $g_B = cg_m$ can occur only when $m = 1$. Conversely, when $m = 1$, Ω_m is the unit ball, and both g_m and g_B are constant multiples of the complex hyperbolic metric. Hence (1) and (4) are equivalent. □

References

[AS83] K. Azukawa and M. Suzuki, *The Bergman metric on a Thullen domain*, Nagoya Math. J. **89** (1983), 1–11.

[Aub78] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. **102** (1978), no. 1, 63–95.

[Cao96] H.-D. Cao, *Existence of gradient Kähler-Ricci solitons*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994) **1** (1996), 16.

[Cao97] ———, *Limits of solutions to the Kähler-Ricci flow*, J. Diff. Geom. **45** (1997), no. 2, 257–272.

[Che] S.-Y. Cheng, *Open problems*, Problem session at the Conference on Nonlinear Problems in Geometry, Katata, Japan, September 3–8, 1979.

[Cho23] B. Chow, *Ricci solitons in low dimensions*, vol. 235, American Mathematical Society, 2023.

[CY80] S.-Y. Cheng and S.-T. Yau, *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math **33** (1980), 507–544.

[Fut83] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), no. 3, 437–443.

[GKK11] R. E. Greene, K.-T. Kim, and S. G. Krantz, *The geometry of complex domains*, vol. 291, Springer Science & Business Media, 2011.

[Gon19] Sébastien Gontard, *Curvatures of metrics in non-compact complex manifolds*, Ph.D. thesis, Université Grenoble Alpes, 2019.

[Ham88] Richard S Hamilton, *The Ricci flow on surfaces, mathematics and general relativity*, Contemp. Math. **71** (1988), 237–261.

[HX21] X. Huang and M. Xiao, *Bergman-Einstein metrics, a generalization of Kerner's theorem and Stein spaces with spherical boundaries*, J. Reine Angew. Math. **2021** (2021), no. 770, 183–203.

[KY96] S. G. Krantz and J. Yu, *On the Bergman invariant and curvatures of the Bergman metric*, Illinois J. Math. **40** (1996), no. 2, 226–244.

[LSY05] K. Liu, X. Sun, and S.T. Yau, *Canonical metrics on the moduli space of Riemann surfaces ii*, J. Differential Geom. **69** (2005), no. 1, 163–216.

[MY83] N. Mok and S.-T. Yau, *Completeness of the Kähler-Einstein metric on bound domains and the characterization of domain of holomorphy by curvature condition*, Proceddings of Symposia in Pure mathematics, V.39 **Part 1** (1983), 41—-59.

[Omo67] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), no. 2, 205–214.

[Seo12] A. Seo, *On a theorem of Paul Yang on negatively pinched bisectional curvature*, Pac. J. Math. **256** (2012), no. 1, 201–209.

[SY94] R.-M. Schoen and S.-T. Yau, *Lectures on differential geometry*, vol. 1, International press Cambridge, MA, 1994.

[Yau75] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), no. 2, 201–228.

[Yau78] ______, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Yeu09] S.-K. Yeung, *Geometry of domains with the uniform squeezing property*, Adv. Math. **221** (2009), no. 2, 547–569.

Zehao Sha, INSTITUTE FOR MATHEMATICS AND FUNDAMENTAL PHYSICS, SHANGHAI, CHINA

Email address: zhsha@imfp.org.cn

Homepage: https://ricciflow19.github.io/