Université Grenoble Alpes

Master 2 Thesis

Scalar Curvature and S^1-valued Harmonic Maps on 3-Manifolds

Author:
Zehao Sha

Supervisor:
Gérard Besson

Thesis of M2R Fundamental Mathematics, Grenoble

Institut Fourier

Abstract

This thesis mainly introduces some relationships between topological properties and scalar curvature on 3-manifolds by using harmonic maps. Principally based on the works of D. Stern and H. Bray, let $u : M^3 \to S^1$ be a non-trivial S^1-valued harmonic map on M, we first explain an estimate relating the scalar curvature to the average Euler characteristics of the level sets $u^{-1}(\theta)$. Then, we use this estimate to see some new proofs of classical results, such as Kronheimer-Mrowka' classification theorem of the Thurston norm, Bray-Brendle-Neves' rigidity theorem for the systole and the T^3 admits no positive scalar curvature metric of Schoen-Yau.

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Contents

Chapter 1

Introduction

In Riemannian geometry, the scalar curvature is the simplest curvature invariant of a Riemannian manifold. Not like the sectional curvature or Ricci curvature, scalar curvature is just a function on the Riemannian manifolds. For each point, it assigns a single real number determined by the intrinsic geometry of the manifold near that point. More precisely, the geometric intuition behind the scalar curvature S is the comparison of volume between a geodesic ball with a sufficiently small radius in given manifolds M^m and the ball in Euclidean space with the same radius, that is,

$$\frac{\mathrm{Vol}\left(B_\varepsilon(p) \subset M\right)}{\mathrm{Vol}\left(B_\varepsilon \subset \mathbb{R}^m\right)} = 1 - \frac{S(p)}{6(m+2)}\varepsilon^2 + O\left(\varepsilon^4\right). \tag{1.1}$$

From this, we can easily to see that, if a manifold M has scalar curvature $S > 0$, then the volume of the geodesic ball with small radius is strictly smaller than the volume of the ball in Euclidean space.

The studies of manifolds with bounded sectional curvature and Ricci curvature give the structure in specific rigid forms. But, those manifolds with scalar curvature bounded below, display an uncertain variety of flexible shapes similar to what we see in geometric topology. Historically, two key techniques have been used to study the global structure of a manifold with bounded scalar curvature, the Dirac operator methods originating in the work of Lichnerowicz [1] (and further developed by Hitchin [2], Witten [3], Gromov-Lawson [4], [5], [6], and others), and the minimal hypersurface methods pioneered by Schoen and Yau [7], [8], [9] in the late 1970s.

In dimension two, the scalar curvature is the twice Gauss curvature, and certainly characterizes the whole curvature of a surface. Thus, we may be interested in the manifolds with bounded scalar curvature for dimension more than two. In [10], Stern introduced the following inequality for S^1-valued harmonic maps on closed 3-manifolds M, relating the scalar curvature to the average Euler characteristic $\chi(\Sigma_\theta)$ of the level sets

$\Sigma_\theta := u^{-1}(\theta)$,

$$\frac{1}{2}\int_{\theta\in S^1}\int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2}+S^M\right)\le 2\pi\int_{\theta\in S^1}\chi(\Sigma_\theta). \tag{1.2}$$

The proof consists of applying Schoen–Yau rearrangement trick to the Ricci term in the Bochner identity combining with the traced Gauss equation. Note that we have to keep away from the case that $\nabla u \neq 0$, so it is necessary to consider a perturbation $(|\nabla u|+\delta)^{1/2}$ for some $\delta > 0$, then taking $\delta \to 0$ at the end of computation. Later, Stern and Bray [11] generalized the previous inequality to the case of compact 3-manifolds with boundary, and we will see these carefully in Chapter 3.

The main application of Stern's inequality is to characterize the Thurston norm of homology class $\alpha \in H^2(M;\mathbb{Z})$ which generalizing the result given by Kronheimer and Mrowka in [12] in terms of the L^2-norm of the negative part of scalar curvature and the harmonic norm of α. Meanwhile, it is also useful of Stern's inequality in the proofs of some classical rigidity result, such as the rigidity of homological 2–systole proved by Bray–Brendle–Neves in [13] and the classical result that T^3 does not have positive scalar curvature metric by Schoen-Yau in [7]. We will discuss these concretely in Chapter 4.

Chapter 2

Foundation for Riemannian Geometry

In this Chapter, we review some notions in Riemannian geometry which we will use later. For more details, one can see, for examples, [14] and [15].

2.1 Riemannian Manifolds

Let (M^m, g) denote a m-dimensional Riemannian manifold with metric g. We will always assume that M is complete with respect to the metric space structure induced by g. Throughout this thesis we use the Einstein convention for summation. In terms of local coordinates $x^1, ..., x^m$, the metric is written in the form

$$g = g_{ij} dx^i dx^j.$$

If $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ are two vector fields, we will also denote their inner product by

$$g(X, Y) := \langle X, Y \rangle = X^i Y^j g_{ij}.$$

Let $\Gamma(TM)$ be the space of smooth vector fields on M. The unique Levi-Civita connection ∇ on M with respect to g is a smooth map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ denoted by $\nabla(X, Y) := \nabla_X Y$, satisfies the following properties:

(1) $\nabla_{(f_1 X_1 + f_2 X_2)} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$;

(2) $\nabla_X (f_1 Y_1 + f_2 Y_2) = X(f_1) Y_1 + f_1 \nabla_X Y_1 + X(f_2) Y_2 + f_2 \nabla_X Y_2$;

(3) $X \langle Y_1, Y_2 \rangle = \langle \nabla_X Y_1, Y_2 \rangle + \langle Y_1, \nabla_X Y_2 \rangle$;

(4) $\nabla_X Y - \nabla_Y X = [X, Y]$,

for all $X, X_1, X_2, Y, Y_1, Y_2 \in \Gamma(TM)$ and $f_1, f_2 \in C^\infty(M)$. The Christoffel symbols Γ_{ij}^k with respect to ∇ are defined by

$$\Gamma_{ij}^k := \nabla_{\partial_i}\partial_j = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

where $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix of metric components (g_{ij}) and ∂_i is the short notation of $\frac{\partial}{\partial x^i}$. Then we have

$$\nabla_X Y = (X^i\partial_i Y^k + X^i Y^j \Gamma_{ij}^k)\partial_k.$$

In a different point of view, given any $X \in \Gamma(TM)$, we may regard $\nabla X : \Gamma(TM \otimes T^*M) \to C^\infty(M)$ as a (1,1)-tensor with the expression

$$\nabla X = (\partial_i X^k + X^j \Gamma_{ij}^k)\partial_k \otimes dx^i.$$

Remark 2.1. We can extend our definition of connection ∇ to any general tensor bundle which satisfies properties (1), (2) and

(5) $\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T$;

(6) $\mathrm{tr}(\nabla_X S) = \nabla_X(\mathrm{tr}\, S)$,

for any $X \in \Gamma(TM)$ and S, T tensor fields, where tr denotes the trace operator.

Next, we introduce some differential operators on M with respect to ∇. For any $f \in C^\infty(M)$, we define the gradient vector field of f by

$$\langle \nabla f, X\rangle := X(f), \quad \text{for all} \quad X \in \Gamma(TM).$$

Locally, we obtain $\nabla f = g^{ij}\partial_i f \partial_j$. Let tr_g^{ij} be the trace operator to tensors in i, j components with respect to g. Note that we can globally define a metric g^{-1} on the cotangent bundle T^*M by $g^{-1} = g^{ij}\partial_i\partial_j$. Then we provide the definition of the divergence of a smooth vector field by

$$\operatorname{div} X := \mathrm{tr}_g^{12}(\nabla X) = \partial_i X^i + X^j \Gamma_{ij}^i.$$

Since we have $\Gamma_{ij}^i = \frac{1}{2}g^{ik}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ji}\right) = \frac{1}{2}g^{ik}\partial_j g_{ik}$ and $\partial_j G = (\partial_j g_{ik})\left(g^{ik}G\right)$, where $G = \det(g_{ij})$. These give another expression of divergence by

$$\operatorname{div} X = \partial_i X^i + \frac{X^i}{\sqrt{G}}\partial_i\sqrt{G} = \frac{1}{\sqrt{G}}\partial_i\left(\sqrt{G}X^i\right).$$

The Laplace-Beltrami operator Δ with respect to g can be given by

$$\Delta f := \operatorname{div}\nabla f = \frac{1}{\sqrt{G}}\partial_i\left(\sqrt{G}g^{ij}\partial_j f\right), \quad \text{for all} \quad f \in C^\infty(M).$$

Finally, for any $X, Y \in \Gamma(TM)$, the Hessian operator $\nabla^2 : C^\infty(M) \to \Gamma(T^*M \otimes T^*M)$ is defined as

$$\nabla^2 f(X,Y) := \langle \nabla_X \nabla f, Y\rangle = XY(f) - \nabla_X Y(f),$$

or in tensor expression

$$\nabla^2 f = (\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f) dx^i \otimes dx^j.$$

Remark 2.2.

1. One can check that $\nabla^2 f : \Gamma(TM) \otimes \Gamma(TM) \to C^\infty(M)$ is a symmetric operator.
2. The Laplace-Beltrami operator can also be defined by the trace of Hessian, i.e. $\Delta f = \mathrm{tr}_g^{12}(\nabla^2 f)$

Example 2.3. If $M^m = \mathbb{R}^m$, then we have $g_{ij} = \delta_{ij}$ and $\Gamma_{ij}^k = 0$. It is easy to check that all differential operator defined above coincide with the usual differential operator in Euclidean space.

The Riemann curvature tensor $R : \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \to C^\infty(M)$ is defined by

$$R(X,Y,Z,W) := \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\rangle,$$

for all $X, Y, Z, W \in \Gamma(TM)$. In local coordinates, we have

$$R_{ijkl} = g_{ln} \left(\partial_i \Gamma_{jk}^n - \partial_j \Gamma_{ik}^n + \Gamma_{jk}^p \Gamma_{ip}^n - \Gamma_{ik}^p \Gamma_{jp}^n \right).$$

Some basic symmetric properties of the Riemann curvature tensor are

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$$

The sectional curvature of the 2-plane $\Pi \subset T_pM$ spanned by any two vector $X_p, Y_p \in T_pM$ is defined by

$$K(\Pi) = K(X_p, Y_p) := \frac{\langle Rm(X_p, Y_p)Y_p, X_p\rangle}{|X_p \wedge Y_p|}.$$

A direct computation show that this equation is independent of the choice of such a basis of Π. In dimension 2, the sectional curvature coincides with the Gauss curvature of a surface at some points.

The Ricci tensor is a symmetric 2-tensor taken by the trace of the Riemann curvature tensor, namely

$$Ric(X,Y) := (tr_g^{14} R)(X,Y),$$

or in tensor expression

$$R_{ij} = g^{kl} R_{kijl}.$$

The scalar curvature is the trace of Ricci tensor,

$$S = g^{ij} R_{ij}.$$

Since the tensor equations are independent of the choice of coordinates, most of the time we would like to choose the normal coordinates centered at some points $p \in M$ to simplify the computation. Recall that, for any $p \in M$, the normal coordinates centered at p carries $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$, where δ_{ij} is the Kronecker symbol.

Theorem 2.4 (Bochner formula)**.** *Let $f \in C^\infty(M)$, we have*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla\Delta f, \nabla f\rangle. \tag{2.1}$$

Proof. We compute this in normal coordinates. Denote ∇_{∂_i} by ∇_i, we have

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla f|^2 &= \nabla_j(\nabla_i f \cdot \nabla_j \nabla_i f) \\
&= (\nabla_j \nabla_i f)^2 + \nabla_j \nabla_j \nabla_i f \cdot \nabla_i f \\
&= |\nabla^2 f|^2 + \nabla_j \nabla_i \nabla_j f \cdot \nabla_i f \\
&= |\nabla^2 f|^2 + \nabla_i \nabla_j \nabla_j f \cdot \nabla_i f + R_{ij} \nabla_i f \cdot \nabla_j f \\
&= |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f) + \langle \nabla\Delta f, \nabla f\rangle.
\end{aligned}$$

This yields (2.1). ☐

Let (Σ^{m-1}, g) be a smooth $(m-1)$-dimensional submanifold of M, with induced metric g from M. The second fundamental form of Σ is a symmetric 2-tensor A, defined by

$$A(X, Y) := \langle \nabla_X \nu, Y\rangle,$$

where ν is the unit normal vector field of Σ. Note that if we choose the other orientation of normal bundle $N\Sigma$, then our definition may be differed by a minus sign. It is well-know that the mean curvature H_Σ of the hypersurface Σ is defined by the trace of second fundamental form, namely,

$$H := \mathrm{tr}_g^{12}(A).$$

Let R^M be the Riemann curvature tensor on M and let R^Σ be the Riemann curvature tensor on Σ. Then,

Lemma 2.5 (Weingarten equation)**.** *Let $\overline{\nabla}$ and ∇ be the Levi-Civita connection on Σ and M respectively. Then for any $X, Y \in \Gamma(T\Sigma)$ and $\nu \in \Gamma(N\Sigma)$, we have*

$$A(X, Y) = -\langle \nu, (\nabla_X Y)^\perp \rangle, \tag{2.2}$$

where $(\nabla_X Y)^\perp$ *denote the projection of vector field* $\nabla_X Y$ *on the normal part that perpendicular to* Σ.

Remark 2.6. Usually, we denote $(\nabla_X Y)^\perp := II(X,Y)$ be the vector second fundamental form on Σ and we have $II(X,Y) = -A(X,Y)\nu \in \Gamma(N\Sigma)$. Obviously, it is symmetric due to the symmetric property of the Levi-Civita connection. For any $X, Y \in \Gamma(T\Sigma)$, after extending them smoothly on M, the vector field $\nabla_X Y$ can be decompose uniquely into the tangential part and the perpendicular part of $T\Sigma$, that is,

$$\nabla_X Y := \overline{\nabla}_X Y + II(X,Y). \tag{2.3}$$

Proof of Lemma 2.5. For any $X, Y \in \Gamma(T\Sigma)$, we extend them smoothly to vector fields on M and still write them by X, Y. So,

$$\begin{aligned} 0 =& X\langle \nu, Y\rangle \\ =& \langle \nabla_X \nu, Y\rangle + \langle \nu, \nabla_X Y\rangle \\ =& \langle \nabla_X \nu, Y\rangle + \langle \nu, \overline{\nabla}_X Y + II(X,Y)\rangle \\ =& A(X,Y) + \langle \nu, II(X,Y)\rangle, \end{aligned}$$

which implies the equation (2.2). □

Theorem 2.7 (Gauss equation)**.** *For any* $X, Y, Z, W \in \Gamma(T\Sigma)$*, we have*

$$R^M(X,Y,Z,W) = R^\Sigma(X,Y,Z,W) - A(X,W)\cdot A(Y,Z) + A(X,Z)\cdot A(Y,W). \tag{2.4}$$

Proof. For any $X, Y, Z, W \in \Gamma(T\Sigma)$, we extend them smoothly to vector fields on M. Then,

$$\begin{aligned} R^M(X,Y,Z,W) =& \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\rangle \\ =& \big\langle \nabla_X \big(\overline{\nabla}_Y Z + II(Y,Z)\big) - \nabla_Y \big(\overline{\nabla}_X Z + II(X,Z)\big) \\ & - \overline{\nabla}_{[X,Y]} Z - II\left([X,Y], Z\right), W\big\rangle. \end{aligned}$$

Since the vector second fundamental form is a normal vector field of Σ, then by the Weingarten equation, we find

$$\begin{aligned} R^M(X,Y,Z,W) =& \big\langle \nabla_X \overline{\nabla}_Y Z - \nabla_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, W\big\rangle \\ & - \big\langle II(X,W), II(Y,Z)\big\rangle + \big\langle II(X,Z), II(Y,W)\big\rangle \\ =& \big\langle \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, W\big\rangle \\ & - A(X,W)\cdot A(Y,Z) + A(X,Z)\cdot A(Y,W) \\ =& R^\Sigma(X,Y,Z,W) - A(X,W)\cdot A(Y,Z) + A(X,Z)\cdot A(Y,W), \end{aligned}$$

which implies the equation (2.4). □

Remark 2.8. In coordinates, we can write (2.4) as

$$R^M_{ijkl} = R^\Sigma_{ijkl} - A_{il}A_{jk} + A_{ik}A_{jl}. \tag{2.5}$$

2.2 Harmonic Maps

Consider a smooth map $u : (M^m, g) \to (N^n, h)$ between two compact Riemannian manifolds with metrics g and h. Let $(x^i)_{i=1}^m$ be a local coordinate near $p \in M$ and let $(y^\alpha)_{\alpha=1}^n$ be a local coordinate near $u(p) \in N$ with $u^\alpha = y^\alpha \circ u$. From now on, to avoid confusion, we use English letters as the indices of the coordinates of M and use Greek letters as the indices of the coordinates of N.

The metrics g and h can be expressed in local coordinates as

$$g = g_{ij}dx^i dx^j \quad \text{and} \quad h = h_{\alpha\beta}dy^\alpha dy^\beta.$$

We define the gradient ∇u as a section of the bundle $TM \otimes u^*TN$, i.e. $\nabla u \in \Gamma(TM \otimes u^*TN)$, where TM is the tangent bundle and u^*TN is the pull-back bundle of TN by u. In local coordinates, we have the expression

$$\nabla u = g^{ik}\partial_k u^\alpha \partial_i \otimes \partial^u_\alpha,$$

where $\partial_i = \frac{\partial}{\partial x^i}$ be the local frame of TM, and $\partial^u_\alpha = (\frac{\partial}{\partial y^\alpha})^u$ be the local frame of u^*TN. The Dirichlet energy density function $e(u) = \frac{1}{2}|\nabla u|^2$ is defined by

$$e(u)(p) = \frac{1}{2}|\nabla u|^2(p) = \frac{1}{2}g^{ij}(p)h_{\alpha\beta}(u(p))\partial_i u^\alpha(p)\partial_j u^\beta(p).$$

Let dV_g be the volume form on M with respect to the metric g. The Dirichlet energy functional is defined by

$$E(u) = \int_M e(u)dV_g. \tag{2.6}$$

Definition 2.9. A smooth map $u : M^m \to N^n$ is said to be harmonic if it is the critical point of the Dirichlet energy functional.

In an intrinsic viewpoint, the differential map $du : TM \to TN$ can be regard as $du = \partial_i u^\alpha dx^i \otimes \partial^u_\alpha \in \Gamma(T^*M \otimes u^*TN)$. Then we see that,

$$e(u) = \frac{1}{2}\langle du, du\rangle_{T^*M\otimes u^*TN},$$

where $\langle \cdot, \cdot \rangle_{T^*M \otimes u^*TN}$ denotes the inner product on $T^*M \otimes u^*TN$ induced from T^*M and u^*TN. Also, we have

$$\mathrm{tr}_g^{12}(u^*h) = g^{ij}(u^*h)(\partial_i, \partial_j) = g^{ij}h(\partial_i u, \partial_j u) = 2e(u).$$

Let ∇ be the covariant derivative on $T^*M \otimes u^*TN$ induced from T^*M and u^*TN. Then we define the hessian of u by $\nabla^2 u := \nabla du \in \Gamma(T^*M \otimes T^*M \otimes u^*TN)$. In local coordinates, we compute that

$$\begin{aligned}
(\nabla^2 u)^\alpha_{ij} &= (\nabla_i du)(\partial_j, dy^\alpha) \\
&= \partial_i \partial_j u^\alpha - du(\Gamma^k_{ij}\partial_k, dy^\alpha) - du(\partial_j, \nabla_i dy^\alpha) \\
&= \partial_i \partial_j u^\alpha - \Gamma^k_{ij}\partial_k u^\alpha - du(\partial_j, \partial_i u^\beta \nabla_\beta dy^\alpha) \\
&= \partial_i \partial_j u^\alpha - \Gamma^k_{ij}\partial_k u^\alpha + \partial_i u^\beta \partial_j u^\gamma (\Gamma^\alpha_{\beta\gamma} \circ u),
\end{aligned}$$

where Γ^k_{ij} and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the metric g and h on M and N respectively. Thus, we have

$$\nabla^2 u = \left(\partial_i \partial_j u^\alpha - \Gamma^k_{ij}\partial_k u^\alpha + \partial_i u^\beta \partial_j u^\gamma (\Gamma^\alpha_{\beta\gamma} \circ u)\right) dx^i \otimes dx^j \otimes \partial^u_\alpha.$$

Definition 2.10. For any smooth map $u : M^m \to N^n$, the map Laplacian of u is defined by

$$\Delta_{g,h} u := \mathrm{tr}_g^{12}(\nabla^2 u) = \left(\Delta u^\alpha + g^{ij}\partial_i u^\beta \partial_j u^\gamma (\Gamma^\alpha_{\beta\gamma} \circ u)\right) \partial^u_\alpha, \tag{2.7}$$

where Δ is the Laplace-Beltrami operator on M with respect to the metric g.

By computing the variational equation of the Dirichlet energy functional, we have the following:

Proposition 2.11. *A smooth map $u : (M^m, g) \to (N^n, h)$ is harmonic if and only if $\Delta_{g,h} u = 0$.*

Proof. For any $\phi \in C^\infty(M, \mathbb{R}^n)$, we consider the variation $u + t\phi$ for sufficient small $|t|$, where u is harmonic. Let G be the determinant of metric (g_{ij}). So we have

$$\begin{aligned}
0 =& \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} \int_M |\nabla u|^2 dV_g \\
=& \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} \int_M g^{ij} h_{\alpha\beta}(u + t\phi)(\partial_i u^\alpha + t\partial_i \phi^\alpha)(\partial_j u^\beta + t\partial_j \phi^\beta)\sqrt{G} dx \\
=& \int_M \left(\frac{1}{2} g^{ij}\partial_\gamma h_{\alpha\beta}(u)\phi^\gamma \partial_i u^\alpha \partial_j u^\beta + g^{ij} h_{\alpha\beta}(u)\partial_i \phi^\alpha \partial_j u^\beta\right)\sqrt{G} dx \\
=& \frac{1}{2}\int_M g^{ij}\partial_\gamma h_{\alpha\beta}(u)\phi^\gamma \partial_i u^\alpha \partial_j u^\beta dV_g \\
& - \int_M \partial_i \left(g^{ij}\partial_j u^\beta \sqrt{G}\right) h_{\alpha\beta}(u)\phi^\alpha + g^{ij}\partial_j u^\beta \partial_i u^\gamma \partial_\gamma h_{\alpha\beta}(u)\phi^\alpha dV_g,
\end{aligned}$$

where the last equality holds because of integration by parts. This implies

$$\begin{aligned}\int_M \Delta u^\beta h_{\alpha\beta}(u)\phi^\alpha dV_g =& \frac{1}{2}\int_M g^{ij}\partial_\gamma h_{\alpha\beta}(u)\phi^\gamma \partial_i u^\alpha \partial_j u^\beta dV_g \\ &- \int_M g^{ij}\partial_j u^\beta \partial_i u^\gamma \partial_\gamma h_{\alpha\beta}(u)\phi^\alpha dV_g.\end{aligned}$$

Taking $\phi^\alpha = h^{\alpha\delta}\eta_\delta$, where $\eta = (\eta_1, ..., \eta_n) \in C^\infty(M, \mathbb{R}^n)$, we obtain

$$\begin{aligned}&\int_M \Delta u^\alpha \eta_\alpha dV_g \\ =& -\frac{1}{2}\int_M g^{ij}h^{\alpha\delta}(u)\left(\partial_\beta h_{\gamma\delta}(u) + \partial_\gamma h_{\beta\delta}(u) - \partial_\delta h_{\beta\gamma}(u)\right)\partial_i u^\gamma \partial_j u^\beta \eta_\alpha dV_g \\ =& -\int_M g^{ij}\partial_i u^\beta \partial_j u^\gamma (\Gamma^\alpha_{\beta\gamma}\circ u)\eta_\alpha dV_g,\end{aligned}$$

which completes the proof, following from (2.7). □

Remark 2.12. Consider a smooth one-parameter family of maps $u_s : (M^m, g) \to (N^n, h)$ with $u_0 = u$. Assume there exists a compact set $K \subset M$, s.t $u_s = u$ on $M \setminus K$ for all s. A similar computation shows the first variation formula

$$\frac{d}{ds}\Big|_{s=0}\int_M e(u_s)dV_g = \int_M \langle \frac{d}{ds}\Big|_{s=0} u_s, \Delta_{g,h}u\rangle dV_g, \tag{2.8}$$

if M is compact without boundary. Or,

$$\frac{d}{ds}\Big|_{s=0}\int_M e(u_s)dV_g = \int_M \langle \frac{d}{ds}\Big|_{s=0} u_s, \Delta_{g,h}u\rangle dV_g + \int_{\partial M}\langle \frac{d}{ds}\Big|_{s=0} u_s, \nabla_\nu u\rangle dV_{g_{\partial M}}, \tag{2.9}$$

if M is compact with boundary, where ν is the unit norm vector field along ∂M.

Example 2.13.
- If $N = \mathbb{R}^n$, then $u : M \to \mathbb{R}^n$ is harmonic if and only if each component of u is a harmonic function on M.
- If $M = S^1$, then, following from the expression of map Laplacian, $u : S^1 \to N$ is harmonic if and only if u is a geodesic in N.

Proposition 2.14. *If $\Phi : (M, g) \to (M, \Phi^* g)$ is a smooth diffeomorphism and $u : (M, g) \to (N, h)$ is a harmonic map. Then $u \circ \Phi : (M, \Phi^* g) \to (N, h)$ is also a harmonic map.*

Proof. In local coordinates, we compute that

$$\begin{aligned}\int_M |\nabla(u\circ\Phi)|^2 dV_{\Phi^* g} &= \int_M g^{ij}h_{\alpha\beta}(u\circ\Phi)\partial_k u^\alpha \partial_i \Phi^k \partial_l u^\beta \partial_j \Phi^l \sqrt{G\circ\Phi}dx \\ &= \int_M g^{ij}h_{\alpha\beta}(u)\partial_i u^\alpha \partial_j u^\beta \sqrt{G}dx \\ &= \int_M |\nabla u|^2 dV_g,\end{aligned}$$

where the second equality we use the coordinates change formula. □

Next, we would like to compute $\Delta_g e(u)$. For simplify, we do the computation in normal coordinate centered at p on M, and the normal coordinate centered at $u(p)$ on N. Let R^M and R^N be the Riemannian curvature tensor on M and N and let Ric^M be the Ricci tensor on M.

Theorem 2.15. *Let $u : (M, g) \to (N, h)$ be a harmonic map. Then we have*

$$\Delta e(u) = |\nabla^2 u|^2 + \left\langle Ric^M, u^* h \right\rangle_g - \operatorname{tr}_g^{24} \operatorname{tr}_g^{13} \left(u^* R^N\right). \tag{2.10}$$

Proof. Let $p \in M$, we compute in normal coordinate at p,

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 =& \frac{1}{2}\Delta\left(g^{ij}h_{\alpha\beta}(u)\partial_i u^\alpha \partial_j u^\beta\right)\\
=& \cancel{g^{kl}h_{\alpha\beta}\partial_i\partial_i\partial_k u^\alpha \partial_l u^\beta} + g^{kl}h_{\alpha\beta}\partial_i\partial_k u^\alpha \partial_i\partial_l u^\beta\\
&- \frac{1}{2}\left(\partial_i\partial_i g_{kl} + \partial_k\partial_l g_{ii} - \partial_i\partial_k g_{il} - \partial_i\partial_l g_{ki}\right) h_{\alpha\beta}\partial_k u^\alpha \partial_l u^\beta\\
&- \frac{1}{2}\left(\partial_\beta\partial_\delta h_{\alpha\gamma} + \partial_\alpha\partial_\gamma h_{\beta\delta} - \partial_\beta\partial_\gamma h_{\alpha\delta} - \partial_\alpha\partial_\delta h_{\beta\gamma}\right) g^{kl}\partial_k u^\alpha \partial_i u^\beta \partial_l u^\gamma \partial_i u^\delta\\
=& |\nabla^2 u|^2 + R^M_{ij}\partial_i u^\alpha \partial_j u^\alpha - R^N_{\alpha\beta\gamma\delta}\partial_i u^\alpha \partial_j u^\beta \partial_i u^\gamma \partial_j u^\delta,
\end{aligned}$$

which is the desired equation. □

Chapter 3

The S^1-valued Harmonic Maps on 3-Manifolds

We will present in this chapter the inequalities given by Stern and Bray in papers [10] and [11] .

3.1 Preliminaries

We first introduce some basic definitions in algebraic topology and Hodge theory for real manifolds. We strongly recommend the readers to look at [14] and [16] for more details. In the rest part of this thesis, we always demand that M is a connected, oriented and compact Riemannian manifold. Let $u : (M^m, g) \to S^1$ be a harmonic map. Consider the pull-back one-form $h := u^* d\theta$ on M. More precisely, set $h = d\tilde{u}$, where $\tilde{u} : M \to \mathbb{R}$ is a local lift of u. Let d be the differential operator to forms and let d^* be the codifferential operator corresponding to d.

Proposition 3.1. *The S^1-valued map u is harmonic if and only if h is a harmonic form, i.e. $dh = 0$ and $d^* h = 0$.*

Proof. Locally, let $h = \partial_i u dx^i$. Compute in normal coordinates, we have

$$dh = d(du) = 0, \quad \text{and} \quad d^* h = -g^{ij} \nabla_i \nabla_j u = -\Delta u,$$

which yields the result. □

Then following from the Hodge theory, we know that every homotopy class $[v]$ in $[M : S^1]$ contains a harmonic representative u such that $u^* d\theta$ is a harmonic 1-form.

By the Hopf classification theorem and Poincaré duality (see [16], p431, Theorem

15 and p297, Theorem 18), we know that each $(m-1)-$homology class α corresponds to a homotopy class $[u] \in [M : S^1]$ whose level sets $\Sigma_\theta := u^{-1}(\theta)$ of some regular points $\theta \in S^1$ represent α.

Remark 3.2. In the case of manifolds with boundary, we recall that by Poincaré-Lefschetz duality and Hopf classification, we then have the isomorphism

$$\left[M : S^1\right] \cong H^1(M;\mathbb{Z}) \cong H_{m-1}(M, \partial M;\mathbb{Z}),$$

such that each $\alpha \in H_{m-1}(M, \partial M;\mathbb{Z})$ can still be represented by the level sets of some non-trivial harmonic maps u.

We now recall the co-area formula and Gauss-Bonnet formula. For simplicity, we only recall the case of scalar-valued functions co-area formula. For further discussion, one can see [17].

Theorem 3.3 (Co-area formula). *Let $u : (M^m, g) \longrightarrow \mathbb{R}$ be a C^1-function and let h be a measurable function on M. Then we have*

$$\int_M h|\nabla u| \; dV_g = \int_{\mathbb{R}} \left(\int_{u^{-1}(x)} h \; dV_g \right) dx. \tag{3.1}$$

Theorem 3.4 (Gauss-Bonnet formula). *Let (M^2, g) be a compact 2-dimensional Riemannian manifold with boundary ∂M. Let K be the Gaussian curvature of M, and let $\boldsymbol{k}$ be the geodesic curvature of ∂M. Then*

$$\int_M K dV_g + \int_{\partial M} \boldsymbol{k} dV_{g_{\partial M}} = 2\pi\chi(M), \tag{3.2}$$

where $\chi(M)$ is the Euler characteristic of M.

Remark 3.5. If M is closed, then we have

$$\int_M K dV_g = 2\pi\chi(M). \tag{3.3}$$

Assume $u : (M^m, g) \to S^1$ is a harmonic map, locally, this is the same thing as a harmonic function, but it is only globally well-defined modulo $\mathbb{Z}$. We consider $\Sigma_\theta := u^{-1}(\theta)$ be the level sets of u for some regular values θ. Then, the gradient vector field ∇u is perpendicular to the level set Σ_θ. We can define $\nu := \frac{\nabla u}{|\nabla u|}$ by the unit normal vector field of Σ_θ. Let A denote the second fundamental form of Σ_θ and let H denote the mean curvature of Σ_θ.

Proposition 3.6. *For any $X, Y \in \Gamma(T\Sigma_\theta)$, we have the following facts,*

1. $A(X,Y) = \frac{\nabla^2 u(X,Y)}{|\nabla u|}$;

2. $|A|^2 = \frac{1}{|\nabla u|^2}\left(|\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + \nabla^2 u(\nu,\nu)^2\right)$;
3. $H^2 = \frac{1}{|\nabla u|^2}\nabla^2 u(\nu,\nu)^2$.

Proof. By the definition of second fundamental form, we have

$$A(X,Y) = \langle \nabla_X \nu, Y\rangle = \frac{1}{|\nabla u|}\langle \nabla_X \nabla u, Y\rangle = \frac{1}{|\nabla u|}\nabla^2 u(X,Y).$$

This prove the first formula.

For the second one, choose an orthonormal basis $\{e_i\}_{i=1}^m$ with $e_m = \nu$, we find

$$\begin{aligned}
|\nabla u|^2|A|^2 &= |\nabla u|^2 \sum_{i,j=1}^{m-1} A(e_i,e_j)^2 \\
&= |\nabla u|^2 \left(\sum_{i,j=1}^{m} A(e_i,e_j)^2 - 2\sum_{i=1}^{m} A(e_i,e_m)^2 + A(e_m,e_m)^2\right) \\
&= |\nabla^2 u|^2 - 2|\nabla^2 u(\cdot,\nu)|^2 + \nabla^2 u(\nu,\nu)^2.
\end{aligned}$$

Also,

$$2\nabla|\nabla u|\cdot|\nabla u| = \nabla|\nabla u|^2 = 2\langle \nabla\nabla u, \nabla u\rangle = 2|\nabla u|\cdot\nabla^2 u(\cdot,\nu).$$

So, we obtain

$$|\nabla u|^2|A|^2 = |\nabla^2 u|^2 - 2|\nabla|\nabla u||^2 + \nabla^2 u(\nu,\nu)^2.$$

Finally, we note that

$$|\nabla u|\cdot H = |\nabla u|\cdot \mathrm{tr}_g^{12}(A) = \mathrm{tr}_g^{12}(\nabla^2 u) - \nabla^2 u(\nu,\nu) = -\nabla^2 u(\nu,\nu),$$

since $\Delta u = 0$. This yields the last equation. □

Lemma 3.7 (Traced Gauss equation)**.** *Let $\Sigma^{m-1}\subset M^m$ be any hypersurface of M and let Ric^M, S^M, S^Σ be the Ricci tensor and scalar curvature on M and Σ respectively. Then.*

$$Ric(\nu,\nu) = \frac{1}{2}\left(S^M - S^\Sigma + H^2 - |A|^2\right). \tag{3.4}$$

Proof. We consider the Gauss equation in local coordinates

$$R^M_{ijkl} = R^\Sigma_{ijkl} - A_{il}A_{jk} + A_{ik}A_{jl}.$$

Taking trace in i,l components with respect to the metric on Σ induced from g, then

$$R^M_{jk} - R^M_{\nu jk\nu} = R^\Sigma_{jk} - HA_{jk} + g^{il}A_{ik}A_{jl}.$$

Then, taking trace in j,k components, we obatin

$$S^M - R^M_{\nu\nu} - R^M_{\nu\nu} = S^\Sigma - H^2 + |A|^2.$$

Rearranging it, we have

$$R^M_{\nu\nu} = \frac{1}{2}\left(S^M - S^\Sigma + H^2 - |A|^2\right),$$

which yields (3.4). □

3.2 Case for Closed 3-Manifolds

In this section, we will show the ideas of Stern's proof of Theorem 1.1 in [10] for closed manifolds.

Theorem 3.8. *Let (M^3, g) be a closed, oriented 3-manifold, and let $u : M \to S^1$ be a non-trivial harmonic map. Then the level sets $\Sigma_\theta := u^{-1}(\theta)$ for some regular values $\theta \in S^1$ of u satisfy*

$$\frac{1}{2}\int_{\theta\in S^1}\int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) \le 2\pi\int_{\theta\in S^1}\chi(\Sigma_\theta). \tag{3.5}$$

Remark 3.9. In general, the level set $\Sigma_\theta := u^{-1}(\theta)$ of S^1-valued harmonic maps u may have several components, and $\chi(\Sigma_\theta)$ denotes the sum of their Euler characteristics.

Proof of Theorem 3.5. Let ∇u be the gradient vector field dual to the harmonic 1-form $u^* d\theta$. By the Bochner formula, we have

$$\Delta|\nabla u|^2 = 2|\nabla^2 u|^2 + 2Ric(\nabla u, \nabla u).$$

Let $\varphi_\delta := (|\nabla u|^2 + \delta)^{1/2}$ for any $\delta > 0$ small enough. We then compute that

$$\Delta\varphi_\delta = \frac{1}{\varphi_\delta}\left(\frac{1}{2}\Delta|\nabla u|^2 - \frac{|\nabla u|^2}{\varphi_\delta^2}|\nabla|\nabla u||^2\right).$$

By Proposition 3.6, we can rewrite the traced Gauss equation

$$Ric(\nabla u, \nabla u) = |\nabla u|^2 Ric(\nu,\nu) = \frac{|\nabla u|^2}{2}\left(S^M - S^\Sigma\right) + |\nabla|\nabla u||^2 - \frac{1}{2}|\nabla^2 u|^2.$$

Then, we have

$$\begin{aligned}
\Delta\varphi_\delta &= \frac{1}{\varphi_\delta}\left(\frac{1}{2}\Delta|\nabla u|^2 - \frac{|\nabla u|^2}{\varphi_\delta^2}|\nabla|\nabla u||^2\right)\\
&\geq \frac{1}{\varphi_\delta}\left(|\nabla^2 u|^2 + Ric(\nabla u, \nabla u) - |\nabla|\nabla u||^2\right)\\
&= \frac{1}{\varphi_\delta}\left(|\nabla^2 u|^2 + \frac{|\nabla u|^2}{2}\left(S^M - S^\Sigma\right) + |\nabla|\nabla u||^2 - \frac{1}{2}|\nabla^2 u|^2 - |\nabla|\nabla u||^2\right)\\
&= \frac{1}{2\varphi_\delta}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right).
\end{aligned}$$

Let $A \subset S^1$ be an open set containing all critical values of u and let $B = S^1 \setminus A$ which is a closed subset of the set of all regular values. Since M is closed, we obtain

$$0 = \int_M \Delta\varphi_\delta = \int_{u^{-1}(A)} \Delta\varphi_\delta + \int_{u^{-1}(B)} \Delta\varphi_\delta.$$

It follows that

$$\int_{u^{-1}(B)} \frac{1}{2\varphi_\delta}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right) \leq -\int_{u^{-1}(A)} \Delta\varphi_\delta.$$

Moreover, since M is compact, there exists a constant $C_M > 0$ depending on the metric structure of M, s.t.

$$\Delta\varphi_\delta \geq \frac{1}{\varphi_\delta}\left(|\nabla^2 u|^2 + Ric(\nabla u, \nabla u) - |\nabla|\nabla u||^2\right) \geq -C_M|\nabla u|.$$

By co-area formula, we then have

$$-\int_{u^{-1}(A)} \Delta\varphi_\delta \leq C_M \int_{u^{-1}(A)} |\nabla u| = C_M \int_A |\Sigma_\theta|,$$

where $|\Sigma_\theta|$ denotes the Hausdorff measure of Σ_θ. Meanwhile, since $\nabla u \neq 0$ on $u^{-1}(B)$, we can pass the limit $\delta \to 0$ and obtain that

$$\int_{u^{-1}(B)} \frac{1}{2|\nabla u|}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right) \leq C_M \int_A |\Sigma_\theta|.$$

In dimension two, the scalar curvature is twice the Gauss curvature. By applying the co-area formula and Gauss-Bonnet formula on the left-hand side, we find

$$\int_{u^{-1}(B)} \frac{1}{2|\nabla u|}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right) = \frac{1}{2}\int_B \int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) - \int_B 2\pi\chi(\Sigma_\theta).$$

Plugging this into the previous estimate gives

$$\frac{1}{2}\int_B \int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) - \int_B 2\pi\chi(\Sigma_\theta) \leq C_M \int_A |\Sigma_\theta|.$$

Finally, by Sard's theorem, we can take the measure of A arbitrarily small, and since $\theta \mapsto |\Sigma_\theta|$ is integrable over S^1 (this is easy to see from the co-area formula). Taking the measure of A goes to 0, we have

$$\int_A |\Sigma_\theta| \longrightarrow 0.$$

Thus, we obtain the desired result. □

Corollary 3.10. *Let (M^m, g) be a closed, oriented m-dimensional manifold, and let $u : M \to S^1$ be a non-trivial harmonic map. Then for any $\phi \in C^\infty(M)$, we have*

$$\frac{1}{2}\int_{\theta\in S^1}\int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right)\phi^2 \le -2\int_M \phi\langle\nabla\phi, \nabla|\nabla u|\rangle. \tag{3.6}$$

Proof. With the same setting in the previous proof, recall that we have the relation

$$\Delta\varphi_\delta \ge \frac{1}{2\varphi_\delta}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right).$$

Multiply any smooth function ϕ^2 in the both side and integrate over $u^{-1}(B)$, we find

$$\begin{aligned}&\int_{u^{-1}(B)}\frac{\phi^2}{2\varphi_\delta}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right)\\ &\le \int_{u^{-1}(B)}\phi^2\Delta\varphi_\delta = \int_{u^{-1}(B)}\left(\Delta(\varphi_\delta\phi) - \varphi_\delta\Delta(\phi^2) - 2\langle\nabla(\phi^2), \nabla\varphi_\delta\rangle\right).\end{aligned}$$

Applying the co-area formula, we obtain

$$\frac{1}{2}\int_B\int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|\varphi_\delta} + S^M - S^{\Sigma_\theta}\right)\phi^2 \le \int_{u^{-1}(B)}\Delta(\varphi_\delta\phi) - \left(\varphi_\delta\Delta(\phi^2) + 2\langle\nabla(\phi^2), \nabla\varphi_\delta\rangle\right)$$

Taking $\delta \to 0$, and for the right-hand side, we note

$$\begin{aligned}\int_{u^{-1}(B)}|\nabla u|\Delta(\phi^2) &= \int_{u^{-1}(B)}|\nabla u|\left(2|\nabla\phi|^2 + 2\phi\Delta\phi\right)\\ &= 2\int_{u^{-1}(B)}|\nabla u||\nabla\phi|^2 - 2\int_{u^{-1}(B)}\langle\nabla(\phi|\nabla u|), \nabla\phi\rangle\\ &= 2\int_{u^{-1}(B)}|\nabla u||\nabla\phi|^2 - 2\int_{u^{-1}(B)}|\nabla u||\nabla\phi|^2 - 2\int_{u^{-1}(B)}\phi\langle\nabla\phi, \nabla|\nabla u|\rangle,\end{aligned}$$

and

$$2\int_{u^{-1}(B)}\langle\nabla(\phi^2), \nabla\varphi_\delta\rangle = 4\int_{u^{-1}(B)}\phi\langle\nabla\phi, \nabla|\nabla u|\rangle.$$

Combining these equations and taking $|A| \to 0$, we obtain the desired inequality. □

From (3.6), it is not hard to see the following facts.

Corollary 3.11. *There does not exist any hypersurface $\Sigma_\theta := u^{-1}(\theta) \subset M$ which can be represented by the level sets of some S^1-valued harmonic maps u, so that*

$$S^M > S^{\Sigma_\theta}.$$

Proof. Assuming there exists a Σ'_θ, such that $S^M > S^{\Sigma'_\theta}$. Then taking $\phi \equiv 1$ in (3.6), we have

$$0 < \int_{S^1}\int_{\Sigma'_\theta}\left(S^M - S^{\Sigma'_\theta}\right) \leq \int_{S^1}\int_{\Sigma'_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma'_\theta}\right) \leq 0,$$

which is a contradiction. □

3.3 Case for Compact 3-Manifolds with Boundary

In this chapter, we will focus on the case of the compact manifolds with boundary. Assume (M^m, g) is a compact Riemannian manifold with boundary ∂M which has the metric induced from M. Recall that from the first variation formula of harmonic maps, a map $u : (M, g) \to S^1$ minimizes the Dirichlet energy in its homotopy class if and only if $du = 0 = d^*u$ and satisfies the homogeneous Neumann boundary condition, i.e.

$$\langle N, \nabla u\rangle = 0,$$

on ∂M, where N is the unit outward normal vector field of ∂M.

Theorem 3.12. *Let (M^3, g) be a compact, oriented 3-manifold with boundary ∂M, and let $u : M \to S^1$ be a non-trivial harmonic map satisfies the homogeneous Neumann boundary condition. Then the level sets $\Sigma_\theta := u^{-1}(\theta)$ for some regular values $\theta \in S^1$ of u satisfy*

$$\frac{1}{2}\int_{\theta\in S^1}\left(\int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) + \int_{\partial\Sigma_\theta} H^{\partial M}\right) \leq 2\pi\int_{\theta\in S^1}\chi(\Sigma_\theta), \tag{3.7}$$

where $H^{\partial M}$ denotes the mean curvature of ∂M.

Proof. Let $\varphi_\delta := (|\nabla u|^2 + \delta)^{1/2}$ for any $\delta > 0$ small enough. With the same computation in Theorem 3.8, we have

$$\Delta\varphi_\delta \geq \frac{1}{2\varphi_\delta}\left(|\nabla^2 u|^2 + |\nabla u|^2\left(S^M - S^\Sigma\right)\right).$$

Away from critical points of u along the boundary ∂M, we see that

$$\begin{aligned}
\langle \nabla \varphi_\delta, N\rangle &= \frac{1}{\varphi_\delta}\Big\langle |\nabla u|\cdot \nabla|\nabla u|, N\Big\rangle \\
&= \frac{1}{2\varphi_\delta}\big\langle \nabla\left(|\nabla u|^2\right), N\big\rangle \\
&= \frac{1}{2\varphi_\delta} N\langle \nabla u, \nabla u\rangle \\
&= \frac{1}{\varphi_\delta}\langle \nabla_N \nabla u, \nabla u\rangle \\
&= \frac{1}{\varphi_\delta}\langle \nabla_{\nabla u}\nabla u, N\rangle \\
&= \frac{1}{\varphi_\delta}\left(\nabla u\cancel{\langle \nabla u, N\rangle} - \langle \nabla u, \nabla_{\nabla u} N\rangle\right).
\end{aligned}$$

Let $A \subset S^1$ be an open set containing all critical values of u and let $B = S^1 \setminus A$ which is a closed subset of the set of all regular values. For some $\theta \in B$, we define

$$Q_\delta(\theta) = \int_{\Sigma_\theta} \frac{|\nabla u|}{2\varphi_\delta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right) + \int_{\partial\Sigma_\theta} \frac{1}{\varphi_\delta}\Big\langle \frac{\nabla u}{|\nabla u|}, \nabla_{\nabla u} N\Big\rangle.$$

By using the co-area formula and combining with the previous computations, we conclude that

$$\begin{aligned}
\int_B Q_\delta(\theta) &= \int_B\int_{\Sigma_\theta} \frac{|\nabla u|}{2\varphi_\delta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right) + \int_B\int_{\partial\Sigma_\theta} \frac{1}{\varphi_\delta}\Big\langle \frac{\nabla u}{|\nabla u|}, \nabla_{\nabla u} N\Big\rangle \\
&= \int_{u^{-1}(B)} \frac{|\nabla u|^2}{2\varphi_\delta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right) + \int_{u^{-1}(B)\cap\partial M} \frac{1}{\varphi_\delta}\langle \nabla u, \nabla_{\nabla u} N\rangle \\
&\le \int_{u^{-1}(B)} \Delta\varphi_\delta + \int_{u^{-1}(B)\cap\partial M} \frac{1}{\varphi_\delta}\langle \nabla u, \nabla_N \nabla u\rangle \\
&\le -\int_{u^{-1}(A)} \Delta\varphi_\delta + \int_{\partial M}\langle \nabla u, N\rangle - \int_{u^{-1}(B)\cap\partial M}\langle \nabla\varphi_\delta, N\rangle \\
&= -\int_{u^{-1}(A)} \Delta\varphi_\delta + \int_{u^{-1}(A)\cap\partial M}\langle \nabla\varphi_\delta, N\rangle \\
&\le C\left(\int_{u^{-1}(A)} |\nabla u| + \int_{u^{-1}(A)\cap\partial M} |\nabla u|\right) \\
&\le C\int_A \left(|\Sigma_\theta| + |\partial\Sigma_\theta|\right),
\end{aligned}$$

where $C > 0$ is a constant.

Let $\nu := \nabla u/|\nabla u|$ be the unit normal vector field of Σ_θ. On the compact set B of regular values, as $\delta \to 0$, we see that Q_δ converges uniformly to

$$Q(\theta) = \int_{\Sigma_\theta} \frac{1}{2}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right) + \int_{\partial\Sigma_\theta}\langle \nu, \nabla_\nu N\rangle.$$

Note that N is also the unit normal vector field of $\partial\Sigma_\theta$ and $N \perp \nu$. We define $\tau := *(\nu \wedge N)$ be the unit vector field tangent to $\partial\Sigma_\theta$. Since ν, τ give an orthonormal basis of ∂M, the mean curvature $H^{\partial M}$ of ∂M is given by

$$H^{\partial M} = A(\nu,\nu) + A(\tau,\tau) = \langle \nu, \nabla_\nu N\rangle + \langle \tau, \nabla_\tau N\rangle.$$

The geodesic curvature of $\partial\Sigma_\theta$ can be defined by $\boldsymbol{k}^{\partial\Sigma_\theta} := \langle \tau, \nabla_\tau N\rangle$. Putting these together and using the Gauss-Bonnet formula, we obtain

$$\begin{aligned} Q(\theta) &= \int_{\Sigma_\theta} \frac{1}{2}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right) + \int_{\partial\Sigma_\theta}\left(H^{\partial M} - \boldsymbol{k}^{\partial\Sigma_\theta}\right) \\ &= \left[\int_{\Sigma_\theta} \frac{1}{2}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) + \int_{\partial\Sigma_\theta} H^{\partial M}\right] - 2\pi\chi(\Sigma_\theta). \end{aligned}$$

Thus, we find

$$\int_B \int_{\Sigma_\theta} \frac{1}{2}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) + \int_B \int_{\partial\Sigma_\theta} H^{\partial M} \le 2\pi \int_B \chi(\Sigma_\theta) + C\int_A \left(|\Sigma_\theta| + |\partial\Sigma_\theta|\right).$$

By Sard's theorem, we may take $|A|$ arbitrarily small. It follows that

$$\int_{S^1} \int_{\Sigma_\theta} \frac{1}{2}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) + \int_{S^1} \int_{\partial\Sigma_\theta} H^{\partial M} \le 2\pi \int_{S^1} \chi(\Sigma_\theta),$$

which yields the desired result. □

Corollary 3.13. *Let (M^m, g) be a compact, oriented m-dimensional manifold with boundary ∂M, and let $u : M \to S^1$ be a non-trivial harmonic map. Then for any $\phi \in C^\infty(M)$, we have*

$$\frac{1}{2}\int_{\theta\in S^1} \int_{\Sigma_\theta} \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right)\phi^2 \le -2\int_M \phi\langle \nabla\phi, \nabla|\nabla u|\rangle + \int_{\partial M} \phi^2 \langle \nabla|\nabla u|, N\rangle. \tag{3.8}$$

Moreover, if $\phi \in C_c^\infty(M)$, then we have

$$\frac{1}{2}\int_{\theta\in S^1} \int_{\Sigma_\theta} \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M - S^{\Sigma_\theta}\right)\phi^2 \le -2\int_M \phi\langle \nabla\phi, \nabla|\nabla u|\rangle. \tag{3.9}$$

Chapter 4

More on Applications

In this chapter, we will show some more applications of Stern's inequalities.

4.1 Characterization of the Thurston Norm

The most important application of Stern's inequalities is to generalize the results shown by Kronhermer and Mrowka in [12].

For a closed, oriented 3–manifold, we recall that the Thurston norm (or semi–norm) of a homology class $\alpha \in H_2(M;\mathbb{Z})$ is defined by the minimum

$$\|\alpha\|_T := \min\{\chi_-(\Sigma) \mid [\Sigma] = \alpha \in H_2(M;\mathbb{Z})\}, \tag{4.1}$$

over all embedded surfaces Σ representing α, where $\chi_-(\Sigma)$ denotes the sum

$$\chi_-(\Sigma) = \max\{0, -\chi(\Sigma_1)\} + \cdots + \max\{0, -\chi(\Sigma_k)\}.$$

of all components $\Sigma_1, ..., \Sigma_k$ of Σ. In 1986, Thurston introduced this concepts in [18] to study the foliations and fibrations of 3–manifolds over S^1.

When the manifold M^3 equips a Riemannian metric, we can define another natural norm on $\alpha \in H_2(M;\mathbb{Z})$, called harmonic norm, by the L^2 norm of the harmonic 1-form $h_\alpha \in \mathcal{H}^1(M)$, with integral periods duals to α, i.e.

$$\|\alpha\|_H := \|h_\alpha\|_{L^2}.$$

In our case, we see that $\|\alpha\|_H = \|\nabla u\|_{L^2}$ which is the L^2 norm of the harmonic map $u : M^3 \to S^1$ whose level sets represent α. It is natural to ask the relation between this two norms, from Stern's inequalities, we have the following truth.

Theorem 4.1. *Let (M^3, g) be a closed, oriented 3–manifold containing no non-separating spheres. Then for any non-trivial class $\alpha \in H_2(M;\mathbb{Z})$, we have*

$$\|\alpha\|_T \leq \frac{1}{4\pi}\|\nabla u\|_{L^2}\|S^-\|_{L^2}, \tag{4.2}$$

where $S^- := \min\{0, -S^M\}$ is the negative part of the scalar curvature of M. If the equality holds for some non-trivial $\alpha \in H_2(M;\mathbb{Z})$, then M is covered isometrically by a cylinder $\Sigma \times \mathbb{R}$ s.t. Σ has constant non-positive curvature.

Proof. Given a nontrivial homology class $\alpha \in H_2(M;\mathbb{Z})$ consider the harmonic map $u : M^3 \to S^1$ whose level sets $\Sigma_\theta = u^{-1}(\theta)$ lie in α. Since M contains no non-separating spheres, it follows every components $\Sigma_1, ..., \Sigma_k$ of Σ_θ must have non-positive Euler characteristics, so that by the definition of the Thurston norm, for all regular values $\theta \in S^1$, we have

$$\|\alpha\|_T \leq -\chi(\Sigma_\theta).$$

Combining this with (3.5), and assuming that the unit circle has length 1, we then obtain

$$\begin{aligned} 2\pi\|\alpha\|_T = \int_{S^1} 2\pi\|\alpha\|_T &\leq -2\pi\int_{S^1}\chi(\Sigma_\theta) \\ &\leq -\frac{1}{2}\int_{S^1}\int_{\Sigma_\theta}\left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M\right) \\ &\leq -\frac{1}{2}\int_{S^1}\int_{\Sigma_\theta} S^M \\ &= -\frac{1}{2}\int_M S^M \cdot |\nabla u|. \end{aligned}$$

The equality holds if and only if Σ_θ attains the minimum of all representatives of α, $|\nabla^2 u| = 0$ and $S^M = S^{\Sigma_\theta}$. By Cauchy-Schwarz inequality, we find

$$2\pi\|\alpha\|_T \leq -\frac{1}{2}\int_M S^M \cdot |\nabla u| \leq \frac{1}{2}\|\nabla u\|_{L^2}\|S^-\|_{L^2},$$

and the equality holds if and only if $S^M = -C|\nabla u|$ for some constant $C > 0$.

In the equality case, we have $\nabla^2 u = 0$ and ∇u is parallel (thus is killing). This implies that the metric splits locally by $\Sigma_\theta \times (-\varepsilon, \varepsilon)$, which is covered isometrically by $\Sigma_\theta \times \mathbb{R}$. □

Now, if we assume that M^3 is compact with boundary, from (3.7), by the same computation, it's not hard to deduce the following fact.

Theorem 4.2. *Let (M^3, g) be a compact, oriented 3–manifold with boundary ∂M such that every connected, embedded surface $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ of genus zero is trivial in*

$H_2(M, \partial M; \mathbb{Z})$. *Then for any non-trivial class* $\alpha \in H_2(M, \partial M; \mathbb{Z})$, *we have*

$$\|\alpha\|_T \leq \frac{1}{4\pi}\|\nabla u\|_{L^2(M)}\|S^-\|_{L^2(M)} + \frac{1}{2\pi}\|\nabla u\|_{L^2(\partial M)}\|H^-\|_{L^2(\partial M)}, \tag{4.3}$$

where $S^- := \min\{0, -S^M\}$ *is the negative part of the scalar curvature of* M *and* H^- *is the negative part of the mean curvature of* ∂M. *If the equality holds for some non-trivial* $\alpha \in H_2(M; \mathbb{Z})$, *then* M *is covered isometrically by a cylinder* $\Sigma \times \mathbb{R}$ *such that* Σ *has constant non-positive curvature and* $\partial\Sigma$ *has constant non-positive geodesic curvature.*

The estimate (4.2) in its dual form, bounding the harmonic norm on $H_2(M)$ above by the product of $\|S^-\|_{L^2}$ and the dual Thurston norm, was proved for irreducible 3-manifolds by Kronheimer and Mrowka in [12]. The proof is very different from this one. And the rigidity result was later proved by Itoh and Yamase in [19]. For more interesting refinements and developments on this topic, we recommend the paper [20] written by Lin.

Note that in Theorem 4.1, we have the assumption that our closed manifolds M contains no non-separating spheres. We now would like to extend this statement to any closed, oriented 3-manifolds. Let $\mathcal{S} \subset H_2(M; \mathbb{Z})$ be a subgroup generated by embedded 2-spheres in M. The first step is to find a collection of disjoint minimal 2-spheres generating $\mathcal{S}$, by using the work of Meeks and Yau [21].

Lemma 4.3. *There exists a finite collection of disjoint embedded, non-separating, minimal 2-spheres* $S_1, ..., S_k \subset M$ *generating* $\mathcal{S}$.

Proof. By Theorems 7 of [21], we know that there exists a finite collection

$$f_1, ..., f_n : S^2 \longrightarrow M$$

of conformal minimal immersions which generate $\pi_2(M)$ as a $\pi_1(M)$-module. Moreover, each map f_i is either an embedding of S^2 or factors through an embedding of $\mathbb{RP}^2$, and the images $f_i(S^2)$ are disjoint.

We consider a sub-collection $f_1, ..., f_k$ so that $(f_i)_*[S^2] \neq 0 \in H_2(M; \mathbb{Z})$ after relabeling. Since the full collection $f_1, ..., f_n$ generates $\pi_2(M)$, by Hurewicz theorem, we know that the push-forwards $S_1 = (f_1)_*[S^2], ..., S_k = (f_k)_*[S^2]$ must generate $\mathcal{S}$ in $H_2(M; \mathbb{Z})$. We claim that f_i doesn't factor through an embedding of $\mathbb{RP}^2$. Otherwise, the non-triviality means that there exists a $\tilde{f} : \mathbb{RP}^2 \to M$ s.t. $(f_i)_*\pi_*[S^2] \neq 0$, where π is the natural projection from S^2 to $\mathbb{RP}^2$. But we have $H_2(\mathbb{RP}^2; \mathbb{Z}) = 0$, that contradicts to the non-triviality assumption. Thus, we have S_i corresponds to an embedded, two-sided 2-sphere in M and $S_i \cap S_j \neq 0$ if $i \neq j$ which satisfies the property we need. $\square$

Now, we cut M along the spheres $S_1, ..., S_k$ of Lemma 4.3 to obtain a compact manifold N with boundary ∂N (see Fig. 4.1). Note that the boundary ∂N of N consists

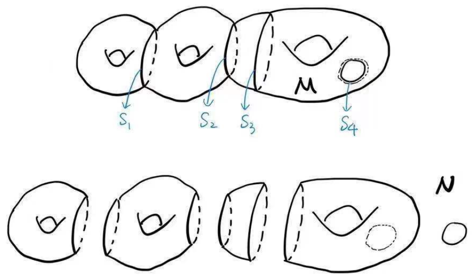

FIGURE 4.1: Cutting M along S_i

of $2k$ minimal spheres, and N carries a local isometry $\Psi : N \to M$ which restricts to a global isometry $N \setminus \partial N \to M \setminus \bigcup_{i=1}^{k} S_i$. We now consider the map $\Phi = \Psi^* : H^1(M;\mathbb{Z}) \to H^1(N;\mathbb{Z})$. We'll see that Φ decreases the harmonic norm.

Lemma 4.4. *For any $\omega \in H^1(M;\mathbb{Z})$, we have*

$$\|\Phi(\omega)\|_H \leq \|\omega\|_H. \tag{4.4}$$

Proof. Let $\omega \in H^1(M;\mathbb{Z}) \cong [M : S^1]$ and let $u : M \to S^1$ be the harmonic representative of ω, so that $\|\omega\|_H = \|\nabla u\|_{L^2}$. The harmonic norm of the class $\Phi(\omega) \in H^1(N;\mathbb{Z})$ is then given by the infimum of the L^2 norm of the gradient among all maps homotopic to $u \circ \Psi$ on N. In particular, by Proposition 2.14, it follows that

$$\|\Phi(\omega)\|_H \leq \|\nabla(u \circ \Psi)\|_{L^2(N)} = \|\nabla u\|_{L^2(M)} = \|\omega\|_H,$$

as claimed. $\square$

Equivalently, by Poincaré-Lefschetz duality, we can also view Φ as a map

$$\Phi : H_2(M;\mathbb{Z}) \to H_2(N, \partial N;\mathbb{Z}).$$

Next, we show that Φ does not decrease the Thurston norm of a class $\alpha \in H_2(M;\mathbb{Z})$. To avoid unnecessary troubles, by using the following simple topological statement, we can remove the boundary term of the representative in $H_2(N, \partial N;\mathbb{Z})$.

Lemma 4.5. *For any connected, embedded surface $(\Sigma, \partial\Sigma) \subset (N, \partial N)$, there exists a closed surface $\tilde{\Sigma} \subset N$ of the same genus such that $[\tilde{\Sigma}] = [\Sigma] \in H_2(N, \partial N;\mathbb{Z})$.*

Proof. Consider a connected, embedded $(\Sigma, \partial\Sigma) \subset (N, \partial N)$, and assume without loss of generality that Σ intersects ∂N transversally along $\partial\Sigma$. Suppose $\partial\Sigma \neq \varnothing$ (otherwise take $\tilde{\Sigma} = \Sigma$), so that $\partial\Sigma$ consists of some number l of embedded loops $\gamma_1, ..., \gamma_l$ in ∂N. Since ∂N consists of spheres, it follows that each boundary component γ_i of Σ may be realized as the boundary of a topological disk $D_i \subset \partial N$. Moreover, since the distinct boundary components γ_i do not intersect, it follows that for every pair D_i, D_j of these disks, either $D_i \cap D_j = \varnothing$, or one disk is strictly contained in the other. In particular, we can find at least one disk D_l, after relabeling, satisfies that for every $i < l$, either $D_l \subset D_i$ or $D_i \cap D_l = \varnothing$.

Attaching this disk D_l to Σ along the corresponding boundary component, we obtain a new (piece-wise smooth) surface Σ' such that the genus $g(\Sigma') = g(\Sigma)$ and has $l-1$ boundary components. We may then perturb Σ' by a homotopy to obtain an embedded surface $\tilde{\Sigma}_l \subset N$ homologous to Σ', meeting ∂N transverally along $\partial\tilde{\Sigma}_l$. Repeating this process, we obtain a sequence $\tilde{\Sigma}_l, ..., \tilde{\Sigma}_1$, terminating at a closed surface $\tilde{\Sigma} = \tilde{\Sigma}_1$ with genus $g(\tilde{\Sigma}) = g(\Sigma)$, such that $[\tilde{\Sigma}] = [\Sigma] \in H_2(N, \partial N; \mathbb{Z})$. □

Lemma 4.6. *For any $\alpha \in H_2(M;\mathbb{Z})$, we have*

$$\|\Phi(\alpha)\|_T \geq \|\alpha\|_T. \tag{4.5}$$

Proof. Consider an embedded surface $\Sigma \subset N$ representing $\Phi(\alpha) \in H_2(N, \partial N; \mathbb{Z})$ such that

$$\chi_-(\Sigma) = \|\alpha\|_T.$$

By Lemma 4.5, we can assume that Σ is closed and doesn't intersect ∂N. Finally, note that we can construct a surface Σ' representing the class $\alpha \in H_2(M;\mathbb{Z})$ by adding some finite combination of the spheres $S_1, ..., S_k$ to the closed representative $\Sigma \subset N \setminus \partial N = M \setminus \bigcup_{i=1}^k S_i$ of $\Phi(\alpha)$. Since the spheres have zero Euler characteristics, we then see that

$$\|\alpha\|_T \leq \chi_-(\Sigma') = \chi_-(\Sigma) = \|\alpha\|_T,$$

as desired. □

Remark 4.7. We observe that any surface of genus 0 gives a trivial class in $H_2(N, \partial N; \mathbb{Z})$. Given any embedded surface $\Sigma \subset N$ of genus zero, we may find a embedded sphere $\tilde{\Sigma} \subset N$ homologous to Σ in $H_2(N, \partial N; \mathbb{Z})$ which must be homologous to some combination of $S_1, ..., S_k$ in M. Thus, $\tilde{\Sigma}$ is homologous to some combination of the components of ∂N, and consequently $[\Sigma] = [\tilde{\Sigma}] = 0 \in H_2(N, \partial N; \mathbb{Z})$.

We now can combine the previous steps to see the following statement.

Theorem 4.8. *Let (M^3, g) be a closed, oriented 3-manifold. Then for any class $\alpha \in H_2(M;\mathbb{Z})$, we have*

$$\|\alpha\|_T \leq \frac{1}{4\pi}\|\alpha\|_H \|S^-\|_{L^2}. \tag{4.6}$$

Moreover, if α cannot be represented by spheres, then equality implies that M is covered isometrically by a cylinder $\Sigma \times \mathbb{R}$ over a closed surface Σ of constant non-positive curvature.

Proof. Given a class $\alpha \in H_2(M;\mathbb{Z})$, consider its image $\Phi(\alpha) \in H_2(N,\partial N;\mathbb{Z})$ under the map $\Phi : H_2(M;\mathbb{Z}) \to H_2(N,\partial N;\mathbb{Z})$ described above. Note that any surface of genus zero in N induces a trivial cycle in $H_2(N,\partial N;\mathbb{Z})$, and since N has minimal boundary, the mean curvature of ∂N is zero. We may apply (4.3) to conclude that

$$4\pi\|\Phi(\alpha)\|_T \le \|\Phi(\alpha)\|_H \|S^-\|_{L^2(N)} = \|\Phi(\alpha)\|_H \|S^-\|_{L^2(M)}.$$

Since Φ decreases the harmonic norm but does not decrease the Thurston norm, we conclude that

$$4\pi\|\alpha\|_T \le \|\alpha\|_H \|S^-\|_{L^2(M)}.$$

Finally, note that if α cannot be represented by spheres, then $\Phi(\alpha) \neq 0$. So that the equality condition holds as same as the previous result. □

Remark 4.9. In [22], Katz has also considered the application of harmonic S^1-valued maps to the study of the Thurston norm of 3–manifolds. The results given by Katz related on the topological features of the maps, rather than their role as a mediator between topology and geometry.

It is not hard to construct a family of metrics on M and giving the following geometric characterization of the Thurston norm.

Corollary 4.10. *Let (M^3, g) be a closed, oriented 3-manifold. Then for any class $\alpha \in H_2(M;\mathbb{Z})$, the Thurston norm is given by the infimum*

$$\|\alpha\|_T = \frac{1}{4\pi} \inf\left\{\|\alpha\|_H \|S^-\|_{L^2} \middle|\ g \in \mathrm{Met}(M)\right\}, \tag{4.7}$$

where $\mathrm{Met}(M)$ *denote the space of all Riemannian metric on M.*

Proof. Fix a non-trivial class $\alpha \in H_2(M;\mathbb{Z})$. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k$ be its representative such that $\|\alpha\|_T = \chi_-(\Sigma)$. Since the sphere components make no contribution in the negative part of Euler characteristic, without loss of generality, after relabeling, we may assume that $\Sigma_1, ..., \Sigma_l$ are the torus components and $\Sigma_l, ..., \Sigma_k$ are the components with negative Euler characteristics.

Let $\delta > 0$ small enough, consider a initial metric $g_{1,\delta}$ on M which coincides on a neighborhood of each Σ_i with the cylinder $\Sigma_i \times [0,1]$, where Σ_i is flat with area δ if $1 \le i \le l$. Moreover, if $l+1 \le j \le k$, Σ_j has constant scalar curvature $S^{\Sigma_j} = -2$ and area

$$|\Sigma_j| = \int_{\Sigma_j} 1 = -\int_{\Sigma_j} K^{\Sigma_j} = 2\pi\chi(\Sigma_j).$$

For $r >> 1$, let $g_{r,\delta}$ be a metric which contains about each Σ_i a product region $T_{r,i} \cong \Sigma_i \times [0,r]$ and coincides with $g_{1,\delta}$ on the complement $E := M \setminus \bigcup_{i=1}^k T_{r,i}$. We then see that

$$\begin{aligned}\int_M \left(S^-_{r,\delta}\right)^2 &= \int_E \left(S^-_{1,\delta}\right)^2 + \int_{\bigcup_{i=1}^k T_{r,i}} \left(S^-_{r,\delta}\right)^2 \\ &= C(\delta) + \sum_{i=1}^k \int_{\Sigma_i} \int_0^r \left(S^-_{r,\delta}\right)^2 \\ &= C(\delta) + 4r \sum_{i=l+1}^k |\Sigma_i| \\ &= C(\delta) - 8\pi r \sum_{i=l+1}^k \chi(\Sigma_i) \\ &= C(\delta) + 8\pi r \|\alpha\|_T.\end{aligned}$$

At the same time, we can define a map $v^r : M \to S^1 = \mathbb{R}/\mathbb{Z}$ in the homotopy class dual to α by

$$v^r(x,t) := \begin{cases} t/r, & \text{for} \quad (x,t) \in T_{r,i} \\ 1, & \text{for} \quad (x,t) \in E. \end{cases}$$

Since the harmonic norm of α is the infimum of the Dirichlet energy go through all harmonic 1-form with integral periods duals to α, We thus compute that,

$$\begin{aligned}\|\alpha\|^2_{H,g_{r,\delta}} \leq \int_M g_{r,\delta}(\nabla v^r, \nabla v^r) &= \sum_{i=1}^k \frac{1}{r^2}|T_{r,i}| \\ &= \sum_{i=1}^l \frac{r\delta}{r^2} - \sum_{i=l+1}^k \frac{r \cdot 2\pi\chi(\Sigma_i)}{r^2} \\ &= \frac{1}{r}\left(l\delta + 2\pi\|\alpha\|_T\right).\end{aligned}$$

Combining with the estimate of the L^2 norm of the scalar curvature, we obtain

$$\begin{aligned}\|\alpha\|^2_{H,g_{r,\delta}} \|S^-_{r,\delta}\|^2_{L^2} &\leq \frac{1}{r}\left(l\delta + 2\pi\|\alpha\|_T\right)\left(C(\delta) + 8\pi r\|\alpha\|_T\right) \\ &= 16\pi^2\|\alpha\|^2_T + 8\pi l\delta\|\alpha\|_T + \frac{C(\delta)(l\delta + 2\pi\|\alpha\|_T)}{r}.\end{aligned}$$

For any fixed $\delta > 0$, taking $r \to \infty$ in the preceding estimate gives

$$\inf\left\{\|\alpha\|_H \|S^-\|_{L^2} |\ g \in \mathrm{Met}(M)\right\} \leq \left(16\pi^2\|\alpha\|^2_T + 8\pi l\delta\|\alpha\|_T\right)^{1/2},$$

and finally taking $\delta \to 0$ gives the desired result

$$\|\alpha\|_T = \frac{1}{4\pi} \inf\left\{\|\alpha\|_H \|S^-\|_{L^2} |\ g \in \mathrm{Met}(M)\right\}.$$

□

4.2 Rigidity result of Homological 2-Systole

On a closed, oriented 3-manifold (M^3, g), we define the homological 2-systole by

$$\mathrm{sys}_2(M) := \inf\left\{|\Sigma^2| \mid \Sigma \subset M \text{ embeded, } [\Sigma] \neq 0 \in H_2(M;\mathbb{Z})\right\}. \tag{4.8}$$

We remark that the estimate (3.5) gives a short different proof of the following rigidity theorem, originally proved by Bray–Brendle–Neves in [13] via the analysis of stable minimal surfaces.

Theorem 4.11. *On a closed, oriented 3–manifold (M^3, g) with positive scalar curvature (PSC for short) and nontrivial homology $H_2(M;\mathbb{Z})$, we have*

$$(\min S^M)\,\mathrm{sys}_2(M) \leq 8\pi, \tag{4.9}$$

with equality only if M is covered isometrically by a cylinder $S^2 \times \mathbb{R}$ over a round sphere.

Before proving this theorem, we first see the following statement immediately from (3.5).

Corollary 4.12. *Let $u : M^3 \to S^1$ be a non-trivial harmonic map on a closed, oriented 3–manifold M with PSC. Then*

$$2\pi \int_{S^1} \chi(\Sigma_\theta) \geq \frac{1}{2}(\min S^M) \int_{S^1} |\Sigma_\theta|, \tag{4.10}$$

where $\Sigma_\theta := u^{-1}(\theta)$ are the level sets of u for some regular values $\theta \in S^1$.

Proof. By (3.5), we directly have

$$\begin{aligned} 2\pi \int_{S^1} \chi(\Sigma_\theta) &\geq \frac{1}{2} \int_{S^1} \int_{\Sigma_\theta} \left(\frac{|\nabla^2 u|^2}{|\nabla u|^2} + S^M \right) \\ &\geq \frac{1}{2}(\min S^M) \int_{S^1} \int_{\Sigma_\theta} 1 \\ &\geq \frac{1}{2}(\min S^M) \int_{S^1} |\Sigma_\theta|, \end{aligned}$$

which yields (4.10). □

Proof of Theorem 4.11. By definition of the homological 2-systole, let $\pi_0(\Sigma_\theta)$ be the connected components of Σ_θ, we then have

$$|\Sigma_\theta| \geq \pi_0(\Sigma_\theta) \cdot \mathrm{sys}_2(M).$$

Meanwhile, we note that $\chi(\Sigma_\theta) \leq \chi(S^2) \cdot \pi_0(\Sigma_\theta) = 2\pi_0(\Sigma_\theta)$. By Corollary 4.10, we obtain

$$\begin{aligned} 4\pi \int_{S^1} \pi_0(\Sigma_\theta) \geq & 2\pi \int_{S^1} \chi(\Sigma_\theta) \\ \geq & \frac{1}{2}(\min S^M) \int_{S^1} |\Sigma_\theta| \\ \geq & \frac{1}{2}(\min S^M) \operatorname{sys}_2(M) \int_{S^1} \pi_0(\Sigma_\theta), \end{aligned}$$

which follows that

$$(\min S^M) \operatorname{sys}_2(M) \leq 8\pi.$$

The equality condition holds if and only if M is covered by $\Sigma \times \mathbb{R}$ isometrically for some surfaces with constant PSC equal to S^M. Since $\chi(\Sigma_i)$ must equal to 2, we know that every components of Σ are round spheres. □

4.3 Estimate of Hessian term

Let (M^m, g) be a closed, oriented m-dimensional Reimannian manifold with $Ric \geq 0$ and the injective radius of M is positive. We now consider a non-trivial S^1-harmonic map $u : M \to S^1$.

On the set of regular points, we may analyze the Hessian term more closely,

$$\nabla|\nabla u|^2 = 2\nabla^2 u(\nabla u, \cdot) \leq 2|\nabla^2 u||\nabla u|.$$

On the other hand, we have

$$\nabla|\nabla u|^2 = 2\nabla|\nabla u| \cdot |\nabla u|.$$

Combining two parts, we obtain the Kato's inequality,

$$|\nabla|\nabla u||^2 \leq |\nabla^2 u|^2. \tag{4.11}$$

Note that this holds for any S^1-valued C^2-map. However, a very important observation is that when u is harmonic, we can improve it, see [23].

Lemma 4.13 (Refined Kato's inequality)**.** *Let $u : M^m \to S^1$ be a harmonic map, then*

$$\left(1 + \frac{1}{m-1}\right) |\nabla|\nabla u||^2 \leq |\nabla^2 u|^2. \tag{4.12}$$

Proof. Choose an orthonormal frame $(e_i)_{i=1}^m$ such that $\nabla u = |\nabla u| e_1$. We then have $\nabla|\nabla u| = \nabla^2 u(e_1, \cdot)$ and $|\nabla|\nabla u||^2 = \sum_{i=1}^m \nabla^2 u(e_1, e_i)^2$. Now, we compute

$$\begin{aligned}|\nabla^2 u|^2 &\geq \sum_{i=1}^m \nabla^2 u(e_1,e_i)^2 + \sum_{i=2}^m \nabla^2 u(e_1,e_i)^2 + \sum_{i=2}^m \nabla^2 u(e_i,e_i)^2 \\ &\geq \sum_{i=1}^m \nabla^2 u(e_1,e_i)^2 + \sum_{i=2}^m \nabla^2 u(e_1,e_i)^2 + \frac{1}{m-1}\left(\sum_{i=2}^m \nabla^2 u(e_i,e_i)\right)^2 \\ &\geq \sum_{i=1}^m \nabla^2 u(e_1,e_i)^2 + \sum_{i=2}^m \nabla^2 u(e_1,e_i)^2 + \frac{1}{m-1}\nabla^2 u(e_1,e_1)^2 \\ &\geq \sum_{i=1}^m \nabla^2 u(e_1,e_i)^2 + \frac{1}{m-1}\sum_{i=1}^m \nabla^2 u(e_1,e_i)^2,\end{aligned}$$

which complete the proof. □

From (3.6), we can see the following truth.

Corollary 4.14. *Let $u : M^m \to S^1$ be a non-trivial harmonic map. For any regular point $p \in M$. Assume that there exists $r > 0$, such that the geodesic ball $B_{2r}(p) = \{q \in M \mid d(p,q) < 2r\}$ contained in the set of all regular points. Then*

$$\int_{B_r(p)} |\nabla^2 u|^2 \leq C_M \int_{B_{2r}(p)} |\nabla u|^2, \tag{4.13}$$

where $C_M > 0$ depends on r and m.

Proof. Consider a cut-off function $\varphi \in C_0^\infty(B_{2r}(p))$, so that

$$\begin{cases} \varphi(x) = 1, & \text{if } x \in B_r(p); \\ 0 \leq \varphi(x) \leq 1, & \text{if } x \in \overline{B_{2r}(p)} \setminus B_r(p); \\ \varphi(x) = 0, & \text{if } x \notin \overline{B_{2r}(p)}; \\ |\nabla \varphi| \leq C/r. \end{cases}$$

Taking $\phi = \varphi |\nabla u|^{1/2}$ and applying co-area formula in (3.6), we obtain

$$\int_{B_{2r}(p)} \left(|\nabla^2 u|^2 + \left(S^M - S^\Sigma\right)|\nabla u|^2\right)\varphi^2 \leq -\int_{B_{2r}(p)} \left(\langle \nabla(\varphi)^2, \nabla(|\nabla u|^2)\rangle + 2\varphi^2 |\nabla|\nabla u||^2\right).$$

From traced Gauss equation, since $Ric \geq 0$, we find

$$\frac{|\nabla u|^2}{2}\left(S^M - S^\Sigma\right) + |\nabla|\nabla u||^2 - \frac{1}{2}|\nabla^2 u|^2 = Ric(\nabla u, \nabla u) \geq 0,$$

which implies

$$|\nabla u|^2 \left(S^\Sigma - S^M\right) \leq 2|\nabla|\nabla u||^2 - |\nabla^2 u|^2 \leq \left(1 - \frac{2}{m}\right)|\nabla^2 u|^2,$$

where the last inequality we use the refined Kato's inequality.

By Cauchy-Schwarz inequality, we compute that

$$\begin{aligned}\frac{2}{m}\int_{B_{2r}(p)}\varphi^2|\nabla^2 u|^2 &\leq 4\int_{B_{2r}(p)}\varphi|\nabla\varphi||\nabla u||\nabla|\nabla u||\\ &\leq \frac{C^2}{\varepsilon r^2}\int_{B_{2r}(p)}|\nabla u|^2+4\varepsilon\int_{B_{2r}(p)}\varphi^2|\nabla|\nabla u||^2\\ &\leq C(\varepsilon,r)\int_{B_{2r}(p)}|\nabla u|^2+4\varepsilon\int_{B_{2r}(p)}\varphi^2|\nabla^2 u|^2.\end{aligned}$$

Taking $\varepsilon > 0$ small enough, we thus obtain

$$\int_{B_r(p)}|\nabla^2 u|^2 \leq \int_{B_{2r}(p)}\varphi^2|\nabla^2 u|^2 \leq C(m,r)\int_{B_{2r}(p)}|\nabla u|^2,$$

which yields (4.13). □

Remark 4.15. For the existence of such cut-off function φ on Riemannian manifolds, one may check, for example, Schoen and Yau's book [24] for details.

4.4 T^3 does not have PSC metric

It is well-known that any metric on T^n of non-negative scalar curvature must be flat. This is of course from the classical results proved by Schoen and Yau [7], or another proof by using Dirac operator methods given by Gromov and Lawson [4].

Here by using Stern's idea, we find the following new proof of this result in the case of three dimension given by Chodosh (see [23]).

Theorem 4.16. *There is no PSC metric on T^3.*

Proof. Assume that (T^3, g) has PSC. Let α be a harmonic representative of $[dx^1]$ and let $u : (T^3, g) \to S^1$ be its non-trivial corresponding harmonic map. Recall the estimate of $\Delta|\nabla u|$ in the proof of Theorem 3.8, we immediately find

$$\begin{aligned}0=\int_{T^3}\Delta|\nabla u| &= \int_{T^3}\frac{1}{2|\nabla u|}(|\nabla^2 u|^2+|\nabla u|^2(S^M-S^\Sigma))\\ &> \int_{T^3}-\frac{1}{2}|\nabla u|S^\Sigma\\ &= \int_{S^1}-2\pi\chi(\Sigma),\end{aligned}$$

where the last equality we use the co-area formula and Gauss-Bonnet formula. We claim that no component of Σ is an embedded sphere. Otherwise, assume Σ' is an embedded sphere component of Σ, then lifting everything to the universal covering and note that $\tilde{u} : (\mathbb{R}^3, \tilde{g}) \to \mathbb{R}$ is $\tilde{g}$-harmonic. By Alexander's theorem (see [25], Theorem 1.1), $\tilde{\Sigma}'$

bounds a ball $B \subset \mathbb{R}^3$. Note that u is constant on Σ', thus we see $\tilde{u}$ is harmonic with constant boundary value in the interior domain of $\tilde{\Sigma}'$. By maximal principle, $\tilde{u}$ (and thus u) is constant everywhere, that's a contradiction.

Thus, no component of Σ is a sphere and $\chi(\Sigma) \leq 0$, which contradicts to the previous inequality. $\square$

Bibliography

[1] André Lichnerowicz. Géometrie différentielle-spineurs harmoniques. *C.R. Acad. Sci. Paris.*, 257(1):7–9, 1963.

[2] Nigel Hitchin. Harmonic spinors. *Adv. Math.*, 14(1):1–55, 1974.

[3] Edward Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.

[4] Mikhael Gromov and Blaine Lawson Jr. Spin and scalar curvature in the presence of a fundamental group. i. *Ann. Math.*, 111(2):209–230, 1980.

[5] Mikhael Gromov and Blaine Lawson Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. Math.*, 111(2):423–434, 1980.

[6] Mikhael Gromov and Blaine Lawson Jr. Positive scalar curvature and the dirac operator on complete riemannian manifolds. *Publ. Math. Inst. Hautes Études Sci.*, 58:83–196, 1983.

[7] Richard Schoen and Shing-Tung Yau. Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature. *Ann. Math.*, 110(1):127–142, 1979.

[8] Richard Schoen and Shing-Tung Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28:159–183, 1979.

[9] Richard Schoen and Shing-Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[10] Daniel Stern. Scalar curvature and harmonic maps to S^1. *to appear in J. Diff. Geom., arXiv preprint, arXiv:1908.09754*, 2019.

[11] Hubert Bray and Daniel Stern. Scalar curvature and harmonic one-forms on three-manifolds with boundary. *to appear in Comm. Anal. Geom., arXiv preprint, arXiv:1911.06803*, 2019.

[12] Peter Kronheimer and Tomasz Mrowka. Scalar curvature and the thurston norm. *Math. Res. Lett.*, 4(6):931–937, 1997.

[13] Hubert Bray, Simon Brendle, and Andre Neves. Rigidity of area-minimizing two-spheres in three-manifolds. *Comm. Anal. Geom.*, 18(4):821–830, 2010.

[14] Jürgen Jost. *Riemannian geometry and geometric analysis, 7th Edition*. Springer, 2017.

[15] Ben Andrews and Christopher Hopper. *The Ricci flow in Riemannian geometry: a complete proof of the differentiable 1/4-pinching sphere theorem*. Springer, 2010.

[16] Edwin Spanier. *Algebraic topology*. Springer Science & Business Media, 1989.

[17] Leon Simon. Introduction to geometric measure theory. *Tsinghua Lectures*, 2014.

[18] William Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 339:99–130, 1986.

[19] Mitsuhiro Itoh and Takahisa Yamase. The dual thurston norm and the geometry of closed 3-manifolds. *Osaka J. Math.*, 43(1):121–129, 2006.

[20] Francesco Lin. Monopole floer homology and the spectral geometry of three-manifolds. *Comm. Anal. Geom.*, 28(5):1211–1219, 2020.

[21] William Meeks and Shing-Tung Yau. Topology of three dimensional manifolds and the embedding problems in minimal surface theory. *Ann. Math.*, 112(3):441–484, 1980.

[22] Gabriel Katz. Harmonic maps $M^3 \to S^1$ and 2-cycles, realizing the thurston norm. *Tran. Amer. Math. Soc.*, 357(3):1177–1224, 2005.

[23] Otis Chodosh. Stable minimal surfaces and positive scalar curvature. *https://web.stanford.edu/~ochodosh/Math258-min-surf.pdf*, 2021.

[24] Richard Schoen and Shing-Tung Yau. *Lectures on differential geometry*, volume 2. International press Cambridge, MA, 1994.

[25] Allen Hatcher. Notes on basic 3-manifold topology. *https://pi.math.cornell.edu/~hatcher/3M/3Mfds.pdf*, 2021.