BLOWING UP A POINT AS A CONNECTED SUM

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Abstract. It is a standard fact that the complex blow-up of a compact complex manifold X at a point is, as a smooth manifold, the connected sum of X with complex projective space. This note gives a self-contained proof of this statement. The aim is to make explicit the bridge between complex geometry and differential topology in a form accessible to newcomers.

1. Introduction

Let X be a compact complex manifold of complex dimension $n \geq 1$, and let $\pi : \widetilde{X} \to X$ be the complex blow-up of X at a point $p \in X$. From the viewpoint of complex geometry, $\widetilde{X}$ is obtained by replacing the point p with the exceptional divisor $E \cong \mathbb{P}^{n-1}$, parametrising complex directions through p. From the viewpoint of differential topology, however, blowing up at a point should be compared with the operation of connected sum: one removes a small ball around p and glues in a suitable piece of another manifold along the resulting spherical boundary.

The purpose of this note is to make this relationship precise and to give a self-contained proof of the following statement:

$$\widetilde{X} \;\cong\; X \# \overline{\mathbb{P}^n}$$

as oriented smooth manifolds, where $\overline{\mathbb{P}^n}$ denotes complex projective space with the opposite orientation. Throughout, all manifolds are assumed to be smooth and oriented, and all complex manifolds are assumed to be without boundary.

2. Blow-up and connected sum

We recall the notions of blow-up at a point and of connected sum of oriented manifolds. We restrict to the case we need: blow-up at a single point and connected sum of manifolds of the same dimension.

2.1. Blow-up of a point in a complex manifold.

There are several equivalent ways to define the blow-up of a point in a complex manifold. For the purposes of this note, it is convenient to use the universal property together with a local model.

Definition 2.1 (Blow-up at a point)**.** Let X be a complex manifold and let $p \in X$. The *blow-up of X at p* is a pair $(\widetilde{X}, \pi)$ consisting of a complex manifold $\widetilde{X}$ and a holomorphic map $\pi : \widetilde{X} \to X$ such that:

- π is a biholomorphism from $\widetilde{X} \setminus E$ onto $X \setminus \{p\}$, where $\mathbb{P}^{n-1} \simeq E := \pi^{-1}(p)$ is a smooth hypersurface (the *exceptional divisor*);
- locally near p, there is a holomorphic chart $\varphi : U \to B^{2n} \subset \mathbb{C}^n$ with $\varphi(p) = 0$ such that $\pi^{-1}(U)$ is biholomorphic to the standard blow-up of B^{2n} at the origin, obtained by blowing up $\mathbb{C}^n$ at 0 and restricting to a compact neighbourhood of the exceptional divisor.

The blow-up $(\widetilde{X}, \pi)$ exists and is unique up to biholomorphism. We write $\widetilde{X} = \mathrm{Bl}_p X$.

In particular, near p the blow-up is modelled on the incidence variety

$$\widetilde{\mathbb{C}^n} := \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \; ; \; z \in \ell\}, \quad \pi(z, \ell) = z,$$

whose exceptional divisor $\pi^{-1}(0)$ is naturally identified with $\mathbb{P}^{n-1}$.

2.2. **Connected sum of oriented manifolds.** We now recall the definition of connected sum of oriented manifolds. We will only need the case of compact manifolds of the same dimension.

Definition 2.2 (Connected sum)**.** Let M and N be compact, connected, oriented smooth manifolds of the same real dimension $m \geq 2$. Choose orientation-preserving embeddings of the closed unit ball $B^m \subset \mathbb{R}^m$ into the interiors of M and N, and denote their images again by $B^m \subset M$ and $B^m \subset N$. Remove the interiors of these balls to obtain manifolds with boundary

$$M^\circ := M \setminus \operatorname{int}(B^m), \qquad N^\circ := N \setminus \operatorname{int}(B^m),$$

each with boundary a sphere S^{m-1}. Choose an orientation-reversing diffeomorphism

$$\phi : \partial M^\circ \longrightarrow \partial N^\circ \cong S^{m-1}.$$

The *connected sum* $M\#N$ is defined to be the oriented smooth manifold obtained by gluing M° and N° along their boundaries via ϕ:

$$M\#N := M^\circ \cup_\phi N^\circ.$$

The resulting oriented diffeomorphism type is independent of all choices up to orientation-preserving diffeomorphism.

We will apply this construction to the case where $M = X$ is a compact complex manifold of complex dimension n, so $m = 2n$, and $N = \overline{\mathbb{P}^n}$ is complex projective n-space endowed with the opposite of its complex orientation.

3. A topological description of the blow-up at a point

We can now state and prove the main theorem, which identifies the blow-up of a compact complex manifold at a point with the connected sum with projective space of opposite orientation.

Theorem 3.1. *Let X be a compact complex manifold of complex dimension $n \geq 1$, and let $\pi : \widetilde{X} \to X$ be the complex blow-up of X at a point $p \in X$. Then $\widetilde{X}$ is diffeomorphic to the connected sum*

$$\widetilde{X} \;\cong\; X \,\#\, \overline{\mathbb{P}^n},$$

where $\overline{\mathbb{P}^n}$ denotes complex projective space with the opposite orientation. The diffeomorphism is canonical up to isotopy.

Proof. We divide the argument into three steps: a local model in $\mathbb{C}^n$, a decomposition of $\mathbb{P}^n$, and a global gluing construction.

Step 1. The blow-up of $\mathbb{C}^n$ at the origin and the bundle $D(\mathcal{O}(-1))$. Consider the complex blow-up of $\mathbb{C}^n$ at the origin,

$$\widetilde{\mathbb{C}^n} := \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \ ; \ z \in \ell\}, \quad \pi(z, \ell) = z.$$

The exceptional divisor $E = \pi^{-1}(0)$ is naturally identified with $\mathbb{P}^{n-1}$. It is standard that $\widetilde{\mathbb{C}^n}$ is diffeomorphic to the total space of the tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}^{n-1}$, where the tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}^{n-1}$ is defined by

$$\mathcal{O}(-1) := \{([\ell], v) \in \mathbb{P}^{n-1} \times \mathbb{C}^n \ ; \ v \in \ell\}, \qquad p([\ell], v) = [\ell].$$

For each $[\ell] \in \mathbb{P}^{n-1}$, the fibre $p^{-1}([\ell])$ is identified with the line $\ell \subset \mathbb{C}^n$.

Fix a Hermitian metric h on $\mathcal{O}(-1)$ and let $D(\mathcal{O}(-1)) := \{([\ell], v) \in \mathcal{O}(-1) \ ; \ h_{[\ell]}(v, v) \leq 1\}$, denote its closed unit disc bundle, and $S(\mathcal{O}(-1)) := \{([\ell], v) \in \mathcal{O}(-1) \ ; \ h_{[\ell]}(v, v) = 1\}$ the corresponding unit sphere bundle. Then $D(\mathcal{O}(-1))$ is a compact manifold with boundary $S(\mathcal{O}(-1))$.

On the other hand, consider a small closed ball $B^{2n} \subset \mathbb{C}^n$ centred at the origin. The blow-up of B^{2n} at 0 is obtained from B^{2n} by replacing the centre with the exceptional divisor. More precisely, there is a diffeomorphism of pairs

$$(\widetilde{B^{2n}}, \partial\widetilde{B^{2n}}) \;\cong\; (D(\mathcal{O}(-1)), S(\mathcal{O}(-1))),$$

and the boundary $S(\mathcal{O}(-1))$ is canonically diffeomorphic to S^{2n-1}. Thus the local model of the blow-up near a point is obtained by removing a small ball and gluing in the disc bundle $D(\mathcal{O}(-1))$ along their common boundary S^{2n-1}.

Step 2. A decomposition of $\mathbb{P}^n$*.* Consider complex projective space $\mathbb{P}^n$ with its Fubini–Study metric. Let $p = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$ and let $H = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n; z_0 = 0\} \cong \mathbb{P}^{n-1}$ be the corresponding hyperplane.

For $\varepsilon > 0$ sufficiently small, the geodesic ball $B^{2n} = B_\varepsilon(p)$ is diffeomorphic to the standard closed ball in $\mathbb{R}^{2n}$, with boundary $\partial B^{2n} \cong S^{2n-1}$. The complement $\mathbb{P}^n \setminus \mathrm{int}(B^{2n})$ is a tubular neighbourhood of H, and by the tubular neighbourhood theorem it is diffeomorphic to the disc bundle of the normal line bundle $N_{H/\mathbb{P}^n} \to H$.

It is known that the normal bundle $N_{H/\mathbb{P}^n}$ is holomorphically isomorphic to the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{P}^{n-1}$. Thus we obtain a diffeomorphism

$$\mathbb{P}^n \setminus \mathrm{int}(B^{2n}) \;\cong\; D(\mathcal{O}(1)),$$

where $D(\mathcal{O}(1))$ denotes the closed unit disc bundle of $\mathcal{O}(1)$ for some choice of Hermitian metric. In particular,

$$\mathbb{P}^n \;=\; B^{2n} \cup_{S^{2n-1}} D(\mathcal{O}(1)).$$

Moreover, since $\mathcal{O}(-1) \simeq \mathcal{O}(1)^*$ and any Hermitian metric induces a fibrewise real-linear isomorphism between a complex line and its dual, the disc bundles $D(\mathcal{O}(1))$ and $D(\mathcal{O}(-1))$ are canonically diffeomorphic as oriented manifolds with boundary. Hence, after reversing orientation, we may (and do) view the complement of a small ball in $\overline{\mathbb{P}^n}$ as the disc bundle $D(\mathcal{O}(-1))$.

Step 3. Global construction on X*.* Let $p \in X$ be the point at which we blow up. Choose a holomorphic chart

$$\varphi : U \longrightarrow B^{2n} \subset \mathbb{C}^n$$

with $\varphi(p) = 0$, and let $B \subset U$ be a small closed ball around p mapped diffeomorphically onto B^{2n}. The blow-up $\widetilde{X}$ is obtained from X by replacing the ball B with the local blow-up $\widetilde{B^{2n}}$ along p. By Step 1, we may identify $\widetilde{B^{2n}}$ with $D(\mathcal{O}(-1))$, glued along their common boundary:

$$\widetilde{X} \;\cong\; (X \setminus \mathrm{int}(B)) \cup_{S^{2n-1}} D(\mathcal{O}(-1)).$$

On the other hand, by Step 2 we have

$$\overline{\mathbb{P}^n} \setminus \mathrm{int}(B^{2n}) \;\cong\; D(\mathcal{O}(-1))$$

as manifolds with boundary, where the boundary is identified with S^{2n-1}. Therefore

$$\widetilde{X} \;\cong\; (X \setminus \mathrm{int}(B)) \cup_{S^{2n-1}} \big(\overline{\mathbb{P}^n} \setminus \mathrm{int}(B^{2n})\big).$$

By definition, the right-hand side is precisely the connected sum $X \# \overline{\mathbb{P}^n}$. The construction depends only on the choice of a small ball around p, and different choices give isotopic identifications. This shows that $\widetilde{X}$ is diffeomorphic to $X \# \overline{\mathbb{P}^n}$, with the diffeomorphism canonical up to isotopy. □

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